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ISO-, GENO-, HYPER-MECHANICS FOR MATTER, THEIR ISODUALS, FOR ANTIMATTER, AND THEIR NOVEL APPLICATIONS IN PHYSICS, CHEMISTRY AND BIOLOGY

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ABSTRACT

Pre-existing numbers, related mathematics and consequential physical theories are generally used for the treatment of new scientific problems. In this memoir we outline the research conducted by various mathematicians, physicists and chemists over the past two decades who have shown that the inverse approach, the construction of new numbers, related new mathematics and consequential new physical theories from open physical, chemical and biological problems, leads to new intriguing formulations of increasing complexity called *iso-, geno- and hyper-mathematics* for the treatment of matter in reversible, irreversible and multi-valued conditions, respectively, plus anti-isomorphic images called *isodual mathematics* for the treatment of antimatter.

These novel formulations are based on new numbers characterized by the lifting of the multiplicative unit of ordinary fields (with characteristic zero) from its traditional value +1 to: (1) invertible, Hermitean and single-valued units for isomathematics; (2) invertible, non-Hermitean and single-valued units for genomathematics; and (3) invertible, non-Hermitean and multi-valued units for hypermathematics; with corresponding liftings of the conventional associative product and consequential lifting of all branches of mathematics admitting a (left and right) multiplicative unit. An anti-Hermitean conjugation applied to the totality of quantities and their operation of the preceding mathematics characterizes the isodual mathematics.

The above new mathematics are then used for corresponding liftings of Newtonian, Hamiltonian and quantum mechanics, today known as *iso-, geno-, hyper-mechanics for the description of matter and their isoduals for antimatter,* with compatible liftings of geometries and symmetries, and, inevitable of contemporary relativities. The above new body of knowledge is also known as *hadronic mechanics, superconductivity and chemistry,* wherein conventional Hamiltonians represent conventional, linear, local and potential interactions among point particles, while generalized units provide an invariant representation of extended, nonspherical and deformable particles under additional nonlinear, nonlocal and nonpotential interactions due to deep mutual penetration of wavepackets at short distances. Whenever the latter effects are ignorable due to large distances, conventional units, mathematics and relativities are recovered identically.

We finally outline the novel scientific and industrial verifications and applications permitted by the new mathematics and relativities in physics, chemistry and biology, including numerous experimental verifications, and applications for new clean energies and fuels which are prohibited by contemporary mechanics, special relativity and related mathematics.

1. INTRODUCTION

Customarily, new problems in physics, chemistry, biology and other quantitative sciences are treated via pre-existing mathematics. Such an approach is certainly valuable at the initiation of new studies. However, with the advancement of scientific knowledge such an approach historically lead to serious limitations and controversies due to the insufficiency of the used mathematics for the problem at hand.

On historical grounds, the above occurrence is illustrated by the fact that the mathematics so effective for the study of planetary systems (Hamiltonian vector fields, Hamilton-Jacobi equations, etc.) resulted to be inadequate for the study of the atomic structure. In fact, the latter mandated the use of a *new mathematics,* that based on infinite dimensional Hilbert spaces over a field of complex numbers, which has no application for planetary mechanics.

Similar occurrences exist in contemporary science due to the continued use for new scientific problems of pre-existing mathematics proved to be so effective in preceding scientific problems. This is the case for:

(1) the lack of a *classical* formulation of antimatter, due to the inapplicability of conventional mathematics so effective for the classical treatment of matter;

(2) the lack of quantitative studies of nonlocal-integral interactions as occurring in chemical valence bonds, due to the inapplicability of conventional mathematics because of its strictly local-differential character;

(3) the lack of representation of the irreversible and multi-valued nature of biological systems, due to insufficiencies of both conventional mathematics and hypermathematics as currently formulated; and other cases.

In this memoir we outline research conducted by numerous mathematicians, physicists and chemists over the past two decades showing that the construction of new mathematics from open scientific problems does indeed permit new, intriguing scientific horizons with far reaching implications in mathematics as well as quantitative science in general.

As we shall see, the emerging new mathematics are based on progressive generalizations of the multiplicative unit $+1$ into everywhere invertible and sufficiently smooth, but otherwise arbitrary quantities (such as numbers, matrices or integro-differential operators) with corresponding generalizations of the associative product, thus implying corresponding generalizations of all branches of conventional mathematics (hereinafter defined as the mathematics based on the multiplicative left and right unit $+1$ over a field of characteristic zero).

In this memoir we also show that the above new mathematics imply certain liftings of Newtonian, Hamiltonian and quantum mechanics with corresponding liftings of geometries and symmetries and, inevitably, of contemporary relativities. In fact, all efforts outlined in this memoir were aimed at the construction of new mechanics today known as *iso-, geno-, and hyper-mechanics* for the description of matter in conditions of increasing complexity, and corresponding *isodual mechanics* for the description of antimatter. All new scientific and industrial applications can then be reduced to a few primitive mechanical or, more properly, relativity axioms.

Alternatively, an objective of this memoir is to show that no truly novel scientific advance is possible without truly novel mechanics. In turn, no mechanics can be considered as truly new without new mathematics. Finally, no mathematics can possibly be truly new without new numbers. This illustrates the reason why, out of the rather vast scientific studies over three decades reviewed in this memoir, primary efforts were devoted to the search of *new numbers* from which new ,mathematics, new relativities and new scientific and industrial applications uniquely follow.

The reader should be aware that the literature in the topic of this memoir is rather vast because it encompasses numerous studies in pure mathematics, applications in various quantitative sciences, several experimental verifications as well as rapidly expanding new industrial applications. As a result, in this paper we can only review the most fundamental aspects of the new formulations. A technical knowledge of the new advances can only be achieved via the study of the quoted literature. To avoid a prohibitive length, references have been restricted to contributions specifically based on the lifting of the unit with a compatible lifting of the product. Regrettably, we have to defer to the specialized literature the treatment of connections with numerous other studies. References are grouped by main fields indicated with square brackets (e.g., [5]), while individual references are indicated with curved brackets (e.g., (201)). Except for monographs, proceedings and reprint volumes, the titles of the individual contributions are not provide to avoid a prohibitive length, as well as because of their lack of general availability in the physics literature without an extensive library search.

The reader should be aware that the new mathematics and their applications are still in their initial stages and so much remains to be done. The author would be grateful for any comment, as well as for the indication by interested colleagues of mathematical or or other references in the *origination* of the new formulations which have escaped his knowledge.

2. CONSTRUCTION OF ISODUAL MECHANICS FROM CLASSICAL ANTIMATTER

2.1: The scientific unbalance caused by antimatter. One of the biggest scientific unbalances of the 20-th century has been the treatment of *matter* at all possible levels, from Newtonian to quantum mechanics, while *antimatter* was solely treated at the level of *second quantization.* In particular, the lack of a consistent *classical* treatment of antimatter left fundamental open problems, such as the inability to study whether a far away galaxy or quasar is made up of matter or of antimatter.

It should be indicated that classical studies of antimatter simply cannot be done by merely reversing the sign of the charge, because of inconsistencies due to the existence of only one quantization channel. In fact, the quantization of a classical particle with the reversed sign of the charge leads to a particle (rather than a charge conjugated antiparticle) with the wrong sign of the charge.

The origin of this scientific unbalance was not of physical nature, and was instead due to the *lack of a mathematics suitable for the classical treatment of antimatter in such a*

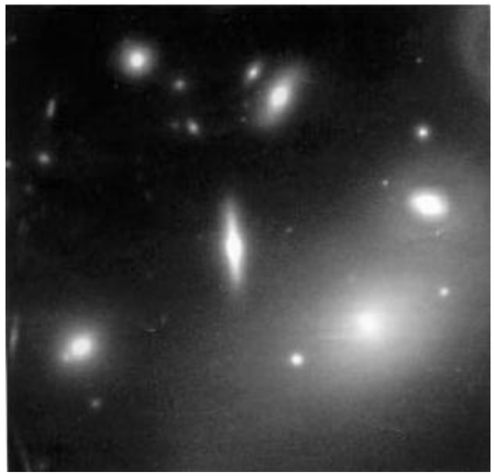

Figure 1: An illustration of the first scientific unbalance of the 20-th century, the inability due to the lack of adequate mathematics to conduct quantitative studies as to whether far away galaxies and quasars are made up of matter or of antimatter.

way to be compatible with charge conjugation at the quantum level. Charge conjugation is an *anti-homomorphism.* Therefore, a necessary condition for a mathematics to be applicable for the classical treatment of antimatter is that of being anti-homomorphic, or, better, anti-isomorphic to conventional mathematics.

The absence of the needed mathematics is confirmed by the fact that classical treatments of antimatter require *fields, functional analysis, differential calculus, topology, geometries, algebras, groups, etc. which are anti-isomorphic to conventional formulations.* The absence in the mathematics of the 20-th century of such a mathematics then mandated its construction as requested by the physical reality here considered (rather than adapting physical reality to pre-existing mathematics).

2.2: Elements of isodual mathematics. A novel mathematics verifying the above conditions was proposed by R. M. Santilli in Ref. (11) of 1985 and then developed in various works (see Refs.(12, 14,15, 54,55,160,161) and [3]).

The fundamental idea is the assumption of a *negative-definite, left and right multiplicative unit,* called *isodual unit,* and denoted I^d, where I denotes the conventional positive-definite unit, $I > 0$,

$$I^d = -I < 0, \tag{2.1}$$

with corresponding reformulation of the conventional associative product $A \times B$ among generic quantities A, B (such as numbers, vector fields, operators, etc.) into the form

$$A \times^d B = A \times (I^d)^{-1} \times B, \tag{2.2}$$

under which I^d is the correct left and right multiplicative unit of the theory,

$$A \times^d I^d = I^d \times^{d^d} A = A, \tag{2.3}$$

for all elements A of the considered set.

More generally, *isodual mathematics is given by the image of a given mathematics admitting a left and right multiplicative unit under the following isodual map*

$$A(x, ...) \rightarrow A^d(x^d, ...) = -A^\dagger(-x^\dagger, ...). \tag{2.4}$$

when applied to the totality of conventional quantities and their operations, with no exception of any type. In this paper we cannot possibly review the entire isodual mathematics, and must restrict ourselves to an elementary review of only the foundations.

DEFINITION 2.1: Let $F = F(a, +, \times)$ be a field of characteristic zero representing real numbers $F = R(n, +, \times), a = n$, complex numbers $F = C(c, +, \times), a = c$, or quaternionic numbers $F = Q(q, +, \times), a = q$, with conventional associative, distributive and commutative sum $a + b = c \in F$, associative and distributive product $a \times b = c \in F$, left and right additive unit 0, $a + 0 = 0 + a = a \in F$, and left and right multiplicative unit $I > 0$, $a \times I = I \times a = a, \forall a, b \in F$. The *isodual fields* (first introduced in Refs. (11,12)) are rings $F^d = F^d(a^d, +^d, \times^d)$ with *isodual numbers*

$$a^d = -a^\dagger, \tag{2.5}$$

associative, distributive and commutative *isodual sum*

$$a^d +^d b^d = -(a + b)^\dagger = c^d \in F^d, \tag{2.6}$$

associative and distributive *isodual product*

$$a^d \times^d b^d = a^d \times (I^d)^{-1} \times b^d = c^d \in F^d, \tag{2.7}$$

additive isodual unit

$$0^d = 0, a^d +^d 0^d = 0^d +^d a^d = a^d, \tag{2.8}$$

and *isodual multiplicative unit*

$$I^d = -I^\dagger, a^d \times^d I^d = I^d \times^d a^d = a^d, \forall a^d, b^d \in F^d. \tag{2.9}$$

LEMMA 2.1 (12): Isodual fields are fields (namely, isodual field verify all axioms of a field with characteristic zero).

The above lemma establishes the property (first identified in Refs. (11,12)) that the axioms of a field *do not* require that the multiplicative unit be necessary positive-definite, because it can also be negative-definite. The proof of the following property is equally simple.

LEMMA 2.2 (12): Fields (of characteristic zero) and their isodual images are anti-isomorphic to each other.

Lemmas 2.1 and 2.2 illustrate the origin of the name "isodual mathematics." In fact, the needed mathematics must constitute a "dual" image of conventional mathematics, while the prefix "iso" is used in its Greek meaning of preserving the original axioms.

It is evident that for real numbers we have $n^d = -n$, while for complex numbers we have $c^d = (n_1 + i \times n_2)^d = -n_1 + i \times n_2 = -\bar{c}$, with a similar formulation for quaternions.

DEFINITION 2.2 (12): A quantity is called *isoselfdual* when it coincides with its isodual.

It is easy to verify that the imaginary unit is isoselfdual because

$$i^d = -\bar{i} = i. \tag{2.10}$$

As we shall see, isoselfduality is a new symmetry with rather profound implications, e.g., in cosmology.

It is evident that, for consistency, *all operations of numbers must be subjected to isoduality.* This implies: the *isodual powers*

$$(a^d)^{n^d} = a^d \times^d a^d \times^d a^d ... \tag{2.11}$$

(n times with n an integer); the *isodual square root*

$$a^{d(1/2)d} = -\sqrt{-a^\dagger}^\dagger, a^{d(1/2)d} \times^d a^{d(1/2)d} = a^d, \quad 1^{d(1/2)d} = -i; \tag{2.12}$$

the *isodual quotient*

$$a^d /^d b^d = -(a^\dagger / b^\dagger) = c^d, \;\; b^d \times^d c^d = a^d; etc. \tag{2.13}$$

An important property for the characterization of antimatter is that *isodual fields have a negative–definite norm*, called *isodual norm* (12)

$$|a^d|^d = |a^\dagger| \times I^d = -(aa^\dagger)^{1/2} < 0, \tag{2.14}$$

where $|...|$ denotes the conventional norm. For isodual real numbers n^d we have the isodual norm $|n^d|^d = -|n| < 0$, the isodual norm for for isodual complex numbers $|c^d|^d = -(n_1^2 + n_2^2)^{1/2}$, etc.

Recall that functional analysis is defined over a field. Therefore, the lifting of fields into isodual fields requires, for necessary condition of consistency, the formulation of the *isodual functional analysis* (54). We here merely recall that

$$\sin^d \theta^d = -\sin(-\theta), \;\; \cos^d \theta^d = -\cos(-\theta), \tag{2.15a}$$

$$\cos^{d\,2d} \theta^d +^d \sin^{d\,2d} \theta^d = 1^d = -1; \tag{2.15b}$$

the *isodual hyperbolic functions*

$$\sinh^d w^d = -\sinh(-w), \; \cosh^d w^d = -\cosh(-w), \tag{2.16a}$$

$$\cosh^{d\,2d} w^d -^d \sinh^{d\,2d} w^d \; = \; 1^d \; = \; -1; \qquad (2.16b)$$

the *isodual logarithm*

$$\log^d n^d \; = \; -log(-n). \qquad (2.17)$$

Particularly important is the *isodual exponentiation* which can be written

$$e^{dA^d} \; = \; I^d + A^d/^d 1!^d + A^d \times^d A^d/^d 2!^d + ... \; = \; -e^{A^\dagger}, \qquad (2.18)$$

Other properties of the isodual functional analysis can be easily derived by the interested reader (see also Refs. (14,21,22).

It is little known that the differential and integral calculi are indeed dependent on the assumed basic unit. In fact, the lifting of I into I^d and of F into F^d implies the *isodual differential calculus*, first introduced in Ref. (14), which is characterized by the *isodual differentials*

$$d^d x^d \; = \; dx, \qquad (2.19)$$

with corresponding *isodual derivatives*

$$\partial^d f^d(x^d)/^d \partial^d x^d \; = \; -\partial \bar{f}(-\bar{x})/\partial(-\bar{x}), \qquad (2.20)$$

and other isodual properties interested readers can easily derive. Note that the differential is isoselfdual.

2.3: Isodual spaces and geometries. Conventional vector and metric spaces are defined over a field. It is then evident that the isoduality of fields requires, for consistency, a corresponding isoduality of vector, metric and all other) spaces.

DEFINITION 2.3: Let $S \; = \; S(x, g, R)$ be an N–dimensional metric space with real-valued local coordinates $x \; = \; \{x^k\}$, $k \; = \; 1, 2, ..., N$, nowhere degenerate, sufficiently smooth, real–valued and symmetric metric $g(x, ...)$ and related line element $x^2 \; = \; (x^t \times g \times x) \times I = (x^i \times g_{ij} \times x^j) \times I \in R$. The *isodual spaces*, first introduced in Refs. (11,14), are vector spaces $S^d(x^d, g^d, R^d)$ with *isodual coordinates* $x^d \; = \; -x^t$ where t denotes transposed, *isodual metric* $g^d(x^d, ...) \; = \; -g^t(-x^t, ...)$, and *isodual line element*

$$(x^d)^{2^d} \; = \; (x^2)^d = (-x^t)^{2d} = (x^d) \times^d (g^d) \times^d (x^{td}) \times^d I^d =$$
$$= [(-x^j)(-\times)(-g_{ji})(-\times)(-x^i)](-\times)(-I) = -x^2 \in R^d. \qquad (2.21)$$

The *isodual Euclidean space* $E^d(x^d, \delta^d, R^d)$ is a particular case of S^d when $g^d_{ij} = \delta^d_{ij}$. The *isodual distance* on E^d is negative definite and it is given by $D^d \; = \; -D$, where D is the conventional (positive-definite) distance on E. The *isodual sphere* on a 3-dimensional isodual space E^d is the perfect sphere with negative radius and expression $r^{d2d} \; = \; [(x_1^{d2d} +^d x_2^{2d} +^d x_3^{2d}] \times^d I^d = -r^2 \in R^d$. The *isodual Minkowskian, isodual Riemannian and isodual symplectic geometries can be defined accordingly* (14,15).

2.4: Isodual Lie theory. Lie's theory in its conventional formulation in mathematics or physics can only characterize matter at the classical level, thus preventing the study of antimatter via fundamental tools so familiar for matter, such as symmetries and conservation laws.

To overcome such an unbalance, R. M. Santilli proposed in the *isodual Lie theory* in the original proposal of isoduality (11), whose explicit form is left to the interested reader. We merely indicate that the isoduality of Lie's theory were proposed also for the classification of all possible realizations of abstract simple Lie algebras.

From the above rudiments interested readers can construct the rest of the isodual mathematics, including: isodual topologies, isodual manifolds, etc. Particularly important for physical applications is the *isodual Lie theory* (first introduced in Ref. (11) (see also (14,22)), including *isodual universal enveloping associative algebras, isodual Lie algebras, isodual Lie groups, isodual symmetries, and isodual representation theory,* which we cannot review here for brevity.

The main physical theories characterized by isodual mathematics can be outlined as follows.

2.5: Isodual Newtonian Mechanics. To resolve the scientific unbalance between matter and antimatter indicated earlier, the isodual mathematics has first permitted a *Newtonian* characterization of antimatter consistent with all available experimental data (14,22). Then, isodual mathematics has identified a new quantization channel (which is distinct from conventional symplectic quantization) leading to an operator formulation which is equivalent to charge conjugation (14,16,21).

We first have the *isodual Newton equations*

$$m^d \times^d \frac{d^d v^d}{d^d t^d} = F^d(t^d, x^d, v^d), v^d = \frac{d^d x^d}{d^d t^d}. \tag{2.22}$$

and related formulations known under the name of *isodual Newtonian Mechanics.*

2.6: Isodual Hamiltonian Mechanics. The direct analytic representation of the above equations are permitted by the following *isodual action functional* (14) (for the case when the Newtonian force F is representable with a potential, see below for nonhamiltonian interactions)

$$\delta^d \mathcal{A}^d(t^d, x^d) = \delta^d \int^d (p^d \times^d d^d x^d -^d H^d \times^d d^d t^d) = 0, \tag{2.23}$$

which characterizes the following the *isodual Hamilton equations* (14)

$$\frac{d^d x^d}{d^d t^d} = \frac{\partial^d H^d}{\partial^d p^d} = p^d, \frac{d^d p^d}{d^d t^d} = -\frac{\partial^d H^d}{\partial^d x^d}, \tag{2.24}$$

with corresponding *isodual Hamilton-Jacobi equations*

$$\frac{\partial^d \mathcal{A}^d}{\partial^d t^d} +^d H^d = 0, \frac{\partial^d \mathcal{A}^d}{\partial^d x^{dk}} -^d p^d_k = 0, k = 1, 2, 3. \tag{2.25}$$

Again, interested readers are encouraged to verify that the above *isodual Hamiltonian Mechanics* does indeed represent correctly all *classical* experimental data on antimatter.

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2.7: Isodual Quantum Mechanics. The isoduality of the naive (or symplectic) quantization can be expressed via the elementary map (14,16,21,55) based on the *isodual Planck's constant* $\hbar^s = I^d = -1$

$$\mathcal{A}^d(t^d, x^d) \rightarrow -i^d \times^d I^d \times^d Ln^d \psi^d(t^d, x^d), \tag{2.26}$$

which can be applied to the isodual Hamilton-Jacobi equations yielding the expressions

$$\frac{\partial^d \mathcal{A}^d}{\partial^d t^d} +^d H^d = 0 \rightarrow -i^d \times^d I^d \times^d \frac{\partial^d \psi^d}{\partial^d t^d} +^d \psi^d \times^d H^d = 0, \tag{2.27a}$$

$$\frac{\partial^d \mathcal{A}^d}{\partial^d x^{dk}} -^d p_k^d = 0 \rightarrow -i^d \times^d I^d \frac{\partial \psi^d}{\partial^d t^d} -^d \psi^d \times^d p_k^d = 0, \tag{2.27b}$$

Therefore, isodual mathematics characterizes the novel *isodual quantum mechanics,* also known as the *isodual branch of hadronic mechanics,* which can be expressed on the *isodual Hilbert space* $\mathcal{H}^d$ with *isodual states* $|\psi>^d = - <\psi^d|$ and *isodual inner product* $<\psi^d| \times^d |\psi^d>$ over the isodual field C^d with basic *isodual Schroedinger equations* (for a Hermitean Hamiltonians H)

$$i^d \times^d <\psi^d| \partial^d /^d \partial^d t^d = - <\psi^d| \times^d H^d = - <\psi^d| \times^d E^d, \tag{2.28a}$$

$$- <\psi^d| \times^d p_k^d = -i^d \times^d <\psi^d| \partial^d /^d \partial^d x^{dk}, \tag{2.28b}$$

and corresponding *isodual Heisenberg equations* (for a Hermitean observable A)

$$i^d \times^d d^d A^d /^d d^d t^d = [A, H]^d = -^d H^d \times^d A^d +^d A^d \times^d H^d = -[A, H], \tag{2.29a}$$

$$[p_i^d, x^{dj}]^d = -i^d \times^d \delta_j^{di}, [x^{di}, x^{dj}]^d = [p_i^d, p_j^d]^d = 0. \tag{2.29b}$$

The equivalence of the above operator formulation of antimatter with charge conjugation has been proved in Refs. (16,21).

2.8: Isodual special relativity. The vast scientific literature on special relativity throughout the 20-th century is silent on the fact that, classically, it can solely represent matter, under the belief that classical antimatter can be represented via a mere change of the sign of the charge without an inspection of the various consequential inconsistencies identified earlier.

To resolve this impasse, R. M. Santilli proposed in Refs. [3] the *isodual special relativity* which is based on the isodual topology (14), the isodual Minkowski space (15), the isodual Poincaré symmetry (29) and the isodualities of relativistic dynamics and physical laws (55). The explicit form of these structures can be easily constructed by the interested reader via the isodual map (2.4).

We only note that that the change of the sign of elementary charges or, more appropriately, charge conjugation, are anti-homomorphic maps, while isoduality is an anti-isomorphic map. Therefore, *according to special relativity antiparticles exist in the same spacetime of particles, while, according to isodual special relativity, antiparticles exist in the isodual spacetime which coexists with, yet it is independent from our spacetime.*

2.9: Origin of the isodual theory in Dirac's equation. On historical grounds it should be indicated that the isodual mathematics and related theory of antimatter originated from an inspection of the celebrated Dirac equation (6). In fact, its basic unit displays precisely the negative-definite unit for the antiparticle component, namely, the isodual unit,

$$\gamma^o = i \times \begin{pmatrix} I_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & -I_{2\times2} \end{pmatrix} = i \times \begin{pmatrix} I_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & I^d_{2\times2} \end{pmatrix} \quad (2.31)$$

Similarly, by recalling that Pauli's matrices are Hermitean, the space components of Dirac's gamma matrices exhibit precisely the isodual Pauli's matrices of antimatter precisely for the antimatter component of the equation,

$$(\gamma^k) = \begin{pmatrix} 0_{2\times2} & \sigma_{2\times2} \\ -\sigma_{2\times2} & 0_{2\times2} \end{pmatrix} = \begin{pmatrix} 0_{2\times2} & -\sigma^d_{2\times2} \\ -\sigma_{2\times2} & 0_{2\times2} \end{pmatrix}. \quad (2.32)$$

It then follows that *Dirac's gamma matrices have the new symmetry of being isoselfdual* (Definition 2.2)

$$\gamma^\mu = -\gamma^{\mu\dagger} = \gamma^{\mu d}, \quad (2.33)$$

and, when interpreted via the isosual mathematics, *Dirac's equation directly describes the Kronecker product of an electron and a positron*

$$\{\gamma^\mu[p_\mu - 2 \times A_\mu)x)/c_o] + i \times m\} \times d\phi(x) =$$

$$= [\begin{pmatrix} 0 & -\sigma^{kd} \\ -\sigma^k & 0 \end{pmatrix} \times (p_k - e \times A_k/c_o) -$$

$$-i \times \begin{pmatrix} I & 0 \\ 0 & I^d \end{pmatrix} \times (p_4 - e \times A_4/c_o) + i \times m] \times \begin{pmatrix} \phi(x) \\ \phi^d(x) \end{pmatrix} = 0. \quad (2.34)$$

Finally, *the true invariance of Dirac's equation is not that under the Poincaré symmetry alone, as popularly believed until now, but rather under the Poincaré symmetry and its isodual*

$$S^{Tot} = P(3.1) \times P^d(3.1). \quad (2.35)$$

Note the elimination of the controversial "hole theory" and second quantization, since the isodual theory of antimatter holds at the *classical* level,, let alone that in first quantization, without excluding, of course, its applicability to second quantization. In particular, as indicated earlier, negative-energy solutions of Dirac's equations mandated the "hole theory" because referred to *positive units,* thus being unphycical, while the negative-energy solutions of Eq. (2.34) are referred in isodual mathematics to *negative units,* thus being fully physical.

Note also the elimination of the additional controversy on the "four-dimensional irreducible representation of spin 1/2" because, under the proper interpretation, Dirac's gamma matrices solely characterize *a two-dimensional representation of spin 1/2,* as a Kronecker product of one representation for matter and its isodual for antimatter. Note the need to eliminate second quantization to admit only two-dimensional irreducible representations of spin 1/2, as mandated by Lie's theory.

Regrettably, Dirac was unaware of the fact that a negative unit can indeed be the correct unit of an appropriate mathematics and, as a consequence, he developed the "hole theory" restricting the treatment of antiparticles to the sole level of second quantization. It is an easy prediction that, in the event the isodual numbers had been discovered prior at the beginning of the 20-th century (rather than in 1985 (11)), Dirac would never have proposed his "hole theory.

2.10: Experimental verifications and applications. It is easy to see that *the isodual theory represents correctly all available classical experimental evidence on antimatter.* For instance, the Coulomb laws for matter with charge q and antimatter with charge q^d at mutual distance r are given by

$$F_{\text{MatterObserver}} = q \times q^d / r^2, \quad F_{\text{AntimatterObserver}} = q^d \times^d q /^d r^{d2}, \tag{2.30a}$$

$$F_{\text{MatterObserver}} = q \times q / r^2, \quad F_{\text{AntimatterObserver}} = q^d \times^d q^d /^d r^{d2}, \tag{2.30b}$$

and they correctly represent mutual attraction (mutual repulsion) for matter-antimatter, (matter-matter and antimatter-antimatter) for both matter and antimatter observers, where F is attractive when having negative value for a matter observer on R, as conventional, and a positive value for an antimatter observer on R^d, the opposite occurring for attraction. For additional details, interested reader can inspect Ref. (22).

The equivalence of the isoduality on Hilbert spaces with charge conjugation proved in Refs. (16,21) establishes that *the isodual theory of antimatter with available experimental data at the operator level too.*

Despite its simplicity, the physical, astrophysical and cosmological implications of isodual mathematics are rather deep. To begin, the isodual map (2.4) implies the change of the sign not only of the charge, but also of all other physical quantities of matter, including mass, energy, time, etc. For instance, the energy eigenvalue E of Eqs. (2.9) has *negative values* since it is positive in Eq. (2.9), yet computed on R^d. Note that the measurement of physical quantities with respect to *negative definite units* resolves the traditional inconsistencies for negative mass and energy.

In particular, the isodual theory of antimatter recovers the old hypothesis that *antiparticles move backward in time* (since they have a negative-definite time) by resolving the inherent violation of causality which lead to its abandonment in the second half of the 20-th century. In fact, *motion backward in time measured with respect to a negative unit of time is as causal as the conventional motion forward in time referred to a positive unit of time.*

Most importantly, *the isodual theory of antimatter mandates the existence of antigravity defined as a gravitational repulsion experienced by antimatter in the field of matter and vice-versa* (16,21,22), while resolving the historical objections against antigravity. As an illustration, the creation of an electron-positron pair is represented by an isoselfdual state on $\mathcal{H} \times \mathcal{H}^d$ over $C \times C^d$, whose eigenvalues can only be defined over the field of the observer (that is, R for a matte observer and R^d for an antimatter observer), as studied in detail in Refs. (21,60). Such an occurrence prevents configurations of correlated electron-positron systems which, for the case of antigravity without the isodual mathematics, could violate the principle of conservation of the energy (e.g., by obtained blueshifts without applied force). Other objections against antigravity for antimatter-matter systems are resolved in similar ways.

J. P. Mills has shown in Ref. (160) that the experimental verification of antigravity proposed in Ref. (17) can be conducted in a resolutory form with available technology. The proposed experiment essentially consists of a *horizontal* vacuum tube of about one meter in diameter and ten meters in length with internal collimators at one end and a scintillator at the other end. The release of photons through the collimators would establish on the scintillator the point of no gravity; the release of electrons would show at the scintillator a downward shift due to gravity ; and the release of positrons would show an upward or downward shift on the scintillator depending on whether they are experiencing antigravity or gravity, respectively. The experiment would be resolutory because, when the electrons and positrons have *very small energy* (of the order of electron Volts), the downward or upward shift of their impact on the scintillator would be visible to the naked eye. This low energy experiment has been ignored by experimentalists in favor of other high energy experiments due to the inconsistencies of the prediction of antigravity when treated with conventional mathematics. It is hoped that, in view of the resolution of these inconsistencies thanks to isodual mathematics, experimentalists will reconsider their view and conduct indeed such a fundamental experiment.

Above all, isodual mathematics has fulfilled the primary scope for which it was constructed, the initiation of quantitative classical studies as to whether far away galaxies and quasars are made-up of matter or of antimatter. In fact, isodual mathematics predicts that *antimatter emits a new photon, called the isodual ;photon (21), which has experimentally detectable characteristics different than those of the ordinary photon emitted by matter*, e.g., the isodual photon is repelled by the gravitational field of our Earth, and has a parity different than that of the ordinary photon. The experimental resolution on Earth whether light from a far away galaxy or quasar is made up of photons or of isodual photons would then resolve the open cosmological problem whether the universe is only made up of matter, or antimatter galaxies and quasars are equally present.

An important application of the isodual theory of antimatter has been developed by J. Dunning Davies (161) who has constructed the first (and only) known *thermodynamics for antimatter stars.* In this way, quantitative cosmological studies on the antimatter component of the universe are already under way.

3. CONSTRUCTION OF ISOMECHANICS FROM NONLOCAL, NONLINEAR AND NONPOTENTIAL INTERACTIONS

3.1: The scientific unbalance caused by nonlocal interactions. Another large scientific unbalance of the 20-th century has been the adaptation of *nonlocal-integral systems* to the pre-existing mathematics which is notoriously *local-differential.*This approach has created serious limitations and controversies which have remained unsolved despite attempts conducted over three quarter of a century, such as:

(1) the lack of numerically exact representations of chemical valence bonds in molecular structures;

(2) the historical inability to achieve an exact representation of nuclear magnetic moments;

(3) the absence of an exact representation of the Bose-Einstein correlation without *ad hoc* free parameters to fit data; and other unresolved problems.

In all these cases we have the mutual overlapping/penetration of particles and/or their wavepackets at distances of the order of $10^{-13}cm$, which conditions are strictly nonlocal-integral and, as such, they are beyond the exact applicability of the mathematical structure, let alone physical laws of quantum mechanics.

It is appropriate here to recall that quantum mechanics and its underlying mathematics permitted a *numerically exact* representation of *all* experimental data of the *Hydrogen atom*. By contrast, the same mathematics and quantum laws have not permitted an equally exact representation of the experimental data of the *Hydrogen molecule,* since a historical 2% of molecular binding energy has been missed for about one century under the rigorous applicability of quantum axioms (thus excluding screenings of the Coulomb law which imply noncanonical/nonunitary transforms, thus exiting the class of equivalence of the original theory).

Since the sole difference between one isolated Hydrogen atom and two atoms coupled into the Hydrogen molecule is given by the electron valence bonds, the above occurrence illustrates the exact validity of quantum mechanics and related mathematics when the systems can be effectively appropriate as point particles at sufficiently large mutual distances (as it is the case for the structure of the Hydrogen atom), while the same theory and related mathematics have a merely approximate character when systems contain interactions at short distances (as it is the case for the mutual overlapping of the wavepackets of valence electrons with antiparallel spins).

The insufficiency is due to the fact that conventional mathematics is *local-differential* in its structure, thus solely permitting the representation of valence bonds as occurring between point particles interacting at a distance. This representation is evidently valid in first approximation because electrons have indeed a point charge. Nevertheless, such a local-differential representation is insufficient because of the lack of treatment of the mutual penetration of the electron wavepackets which cannot be consistently reduced to a finite set of isolated points.

It is then evident that a more adequate treatment of valence bonds in chemistry, as well as all nonlocal-integral interactions in general, requires a *new mathematics* which is partly local-differential (to represent conventional Coulomb interactions) and partly *nonlocal-integral* (to represent the overlapping of the wavepackets). Additional physical requirements establish that the needed mathematics must be *nonlinear in the wavefunction*, thus preventing the use of conventional quantum mechanics because nonlinear Schroedinger's equations violate the superposition principle with consequential inapplicability to composite systems (45,46).

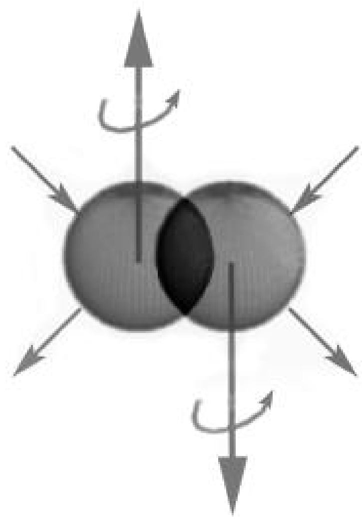

Figure 2: An illustration of the second scientific unbalance of the 20-th century, the lack of adequate mathematics for quantitative studies of nonlocal-integral interactions occurring the in deep overlappings of the wavepackets of electron valence bonds in molecules, as well as, in general, all interactions of particles ar short distances.

It should be indicated that numerous "nonlocal interactions" exist in the physical literature, but they have generally been adapted to be representable with a local potential in a Hamiltonian (for theoretical and experimental studies on this type of nonlocality nonlocality, see, e.g., C. A. C. Dreismann (220-223) and references quoted therein).

The interactions occurring in valence bonds are dramatically more general than the above inasmuch as they occur in the finite volume of wave overlappings which is not reducible to a finite number of isolated points and they are of *nonpotential* type because of contact, zero-range nature, thus prohibiting a consistent representation with a Hamiltonian, let alone a potential. More specifically, the granting of a potential energy to the deep wave-overlappings of valence bonds would be the same as granting potential energy to the resistive force experienced by an extended object moving within a medium.

All in all, one century of failed attempt to achieve an exact representation of molecular data establish beyond credible doubts that the nonlocal, nonlinear and nonpotential interactions occurring in valence bonds are beyond the axiomatic structure of quantum mechanics, thus demanding *nonunitary theories* as a necessary condition to exit from the class of unitary equivalence of quantum mechanics.

3.2: Catastrophic inconsistencies of noncanonical-nonunitary theories. Recall that a crucial feature of quantum mechanics, which permitted its dominance in physical applications, is the *invariance,* namely, the preservation of numerical values, physical laws and mathematical axioms under the time evolution of the theory. Since all quantum time evolutions are expressible via a Hermitean Hamiltonian, the above feature is essentially expressed by the invariance of the mathematical structure of quantum mechanics under *unitary* transforms.

In fact, quantum theories are expressible via a Hermitean Hamiltonian $H = p^2/2m + V = H^\dagger$ defined on a Hilbert space $\mathcal{H}$ over the field of complex number C with a *unitary time evolution*

$$U(t) \times |\psi(t_o) >= |\psi(t) >, U \times U^\dagger = U^\dagger \times U = I, U(t) = e^{i \times H \times t}, \tag{3.1}$$

which, as such, leaves invariant all basic units by definition,

$$I \to I' = U \times I \times U^\dagger = I, \tag{3.2}$$

the conventional associative product $A \times B$ between two arbitrary quantities A and B (numbers, matrices, operators, etc.)

$$A \times B \to U \times (A \times B) \times U^\dagger = (U \times A \times U^\dagger) \times (U \times B \times U^\dagger) = A' \times B', \tag{3.3}$$

and all numerical predictions. To illustrate this important property, suppose that a given quantum model; predicts the energy of 5 eV according to the familiar equation

$$H(t_o) \times |\psi(t_o) >= 5eV \times |\psi(t_o) > . \tag{3.4}$$

Then the value 5 eV is preserved at all subsequent times in view of the unitary invariance

$$U(t) \times H(t_o) \times |\psi(t_o) >= (U \times H \times U^\dagger) \times (U \times U^\dagger)^{-1} \times (U \times |\psi >) =$$

$$= H(t) \times |\psi(t) >= U(t) \times (5eV \times |\psi(t_o) >) = 5eV \times |\psi(t) > . \tag{3.5}$$

The invariance of physical laws and mathematical axioms then follows.

It is evident that no representation of, the interactions occurring the valence bonds, and other mutual penetrations of particles at short distances, is physically acceptable unless it achieves the same invariance enjoyed by quantum mechanics. But the interactions herein considered are assumed to be *non-Hamiltonian,* thus being*nonunitary* by conception. The difficulties of the task at hand can then be expressed by the following

THEOREM 3.1 (45,46): All formulations with classical noncanonical and operator nonunitary time evolutions do not have time-invariant numerical predictions, physical laws and mathematical axioms when formulated via the mathematics of classical and quantum Hamiltonian mechanics, thus having no known physical or mathematical value.

To illustrate the above theorem, note that, by assumption, the time evolution needed to represent nonlocal, nonpotential and nonlinear valence bonds has the nonunitary structure

$$W(t) \times |\phi(t_o) >= |\psi(t) >, W \times W^\dagger \neq I, W(t) \neq e^{i \times H \times t}. \tag{3.6}$$

It is then easy to see that, *when the nonunitary theory is formulated via the mathematics of unitary theories, it does not admit time-invariant numerical predictions, thus having no known physical meaning or value* (46).

To illustrate this important occurrence, note that *nonunitary time evolutions do not preserve the basic units by conception,*

$$I \rightarrow I' = W \times I \times W^\dagger \neq I, \tag{3.7}$$

where one should keep in mind that a realization of I is given by the basic units of the Euclidean space $I = Diag.(1cm, 1cm, 1cm)$ expressed in the usual dimensionless form $I = diag.(1, 1, 1)$..

Similarly, *nonunitarity transforms do not preserve the conventional associative product,*

$$A \times B \rightarrow W \times (A \times B) \times W^\dagger = (W \times A \times W^\dagger) \times (W \times W^\dagger)^{-1} \times (W \times B \times W^\dagger) =$$

$$= A' \times (W \times W^\dagger)^{-1} \times B' \neq A' \times B'. \tag{3.8}$$

It is then easy to verify the lack of invariance of numerical, predictions. In fact, suppose that the given nonunitary theory also predicts the energy of 5 eV at the initial time,

$$H(t_o) \times |\psi(t_o)> = 5eV \times |\psi(t_o)>, \tag{3.9}$$

and that 15 seconds later

$$(W \times W^\dagger)_{t=15sec} = 1/3. \tag{3.10}$$

We then have

$$W \times H(t_o) \times |\psi(t_o)> = (W \times H \times W^\dagger) \times (W \times W^\dagger)^{-1} \times (W \times |\psi>) =$$

$$= W(t) \times (5eV \times |\psi(t_o)>), \tag{3.11}$$

namely,

$$H(t) \times |\psi(t)> = (W \times W^\dagger) \times (5eV|\psi(t)>) = 15eV \times |\psi(t)> . \tag{3.12}$$

The lack of preservation in time of numerical predictions then implies consequences today known as "catastrophic physical inconsistencies," such as: the lack of applicability of the theory to measurements (because of the loss of invariant basic units); Lopez's Lemma (172,173) (loss of Hermiticity in time with consequential lack of acceptable observables); violation of causality and probability laws; etc. (45,46,171-175).

The use of conventional mathematics for broader nonunitary theories leads to equally serious mathematical problems today known as "catastrophic mathematical inconsistencies." Suppose that a nonunitary theories is formulated at a given initial time t_o on a metric space defined over the field of real numbers R with basic unit I. But, by their very definition, nonunitary time evolutions do not leave invariant the basic unit, $I \rightarrow I' = U \times I \times U^\dagger \neq I$. Therefore, nonunitary time evolutions do not admit the original basic unit I at all times $t > t_o$, with consequential loss of the base field R. In turn, the loss of the base field implies the inability to properly define the metric space acting on it, with consequential catastrophic collapse of the entire mathematical structure. Classical noncanonical theories are afflicted by similar catastrophic mathematical and physical inconsistencies (45,46,171-175).

It should be recalled that, when facing non-Hamiltonian systems, a rather general tendency is that of transforming such systems into a form which is (at least locally) Hamiltonian via the use of Darboux's theorem of the symplectic geometry or the Lie-Koening theorem of analytic mechanics (51). Unfortunately, these transformations cannot be used for the study of valence bonds because:

(i) the system considered are nonlocal (thus implying the inapplicability *a priori* of the topologies needed for the quoted theorems);

(ii) Darboux's transformations are nonlinear, thus implying the impossibility of placing measuring apparata in the new Darboux's coordinates; and

(iii) in view of their nonlinearity, Darboux's transformations cause the loss of the inertial character of reference frames with consequential loss of Galileo's and Einstein's relativities.

In view of these shortcomings, the only physically acceptable representations are those occurring in the fixed coordinates of the observer, called *direct representations* (51). Only after such a representation is consistently achieved the transformation theory may have value.

In summary, the representation of chemical bonds as well as other nonlocal interactions at short distances requires the abandonment of Hamiltonian theories which, in turn, implies the necessary use of theories whose time evolution is noncanonical at the classical level and nonunitary at the operator level. Still in turn, such noncanonical/nonunitary theories require the necessary construction of a new mathematics capable of resolving the catastrophic inconsistencies reviewed above.

3.3: Elements of isomathematics. The new mathematics specifically constructed for quantitative invariant treatment of nonlocal, nonpotential and nonlinear interactions among extended particles under mutual penetration at short distance is today known under the name of *isomathematics,* (where the prefix "iso" also denotes the preservation of conventional axioms). Isomathematics was first proposed by R. M. Santilli in Ref. (23) of 1978 and subsequently studied by several mathematicians, theoreticians and experimentalists (see Refs. [2,4-11]).

The main idea is that, as indicated earlier, valence bonds include conventional local-differential Coulomb interactions, plus nonlocal, nonlinear and nonpotential interactions due to wave-overlappings. The former interactions can be effectively represented with the conventional Hamiltonian, while the latter interactions can be represented via a generalization of the basic unit as a condition to achieve invariance (since the unit is the basic invariant of any theory).

Therefore, the main assumption of isomathematics is the lifting of the conventional unit of current formulations, generally given by an N-dimensional unit matrix $I = \text{Diag.}(1, 1, ..., 1) > 0$, into a quantity $\hat{I}$, called *isounit,* which possesses all topological properties of I (such a positive-definiteness, same dimensionality, etc.), while having an arbitrary functional dependence on coordinates x, velocities v, wavefunctions ψ, their derivatives $\partial_x\psi$, and any other needed variable (23,51),

$$I = Diag.(1, 1, ..., 1) > 0 \rightarrow \hat{I}(x, v, \psi, \partial_x\psi, ...) = 1/\hat{T}(x, v, \psi, \partial_x\psi, ...) > 0. \quad (3.13)$$

The conventional associative and distributive product $A \times B$ among generic quantities A, B (such as numbers, vector fields, operators, etc.) is jointly lifted into the more general form

$$A \times B \rightarrow A\hat{\times}B = A \times \hat{T} \times B, \quad (3.14)$$

which remains associative and distributive, thus being called *isoproduct,* under which the left and right character of I is preserved, i.e.,

$$I \times A = A \times I = A \rightarrow \hat{I}\hat{\times}A = A\hat{\times}\hat{I} = A, \quad (3.15).$$

for all elements A of the set considered.

An example of isounits (hereon assumed to be multiplicative) representing the interactions due to wave-overlappings of valence bonds is given by

$$\hat{I} = e^{N(\psi)\times\int dx \times \psi^{\dagger}(x)\times\psi(x)}, \quad (3.16)$$

and illustrates the desired representation of nonlocal, nonlinear and nonpotential interactions.

Another example of isounits is given by

$$\hat{I} = Diag.(n_1^2, n_2^2, n_3^2), \quad (3.17)$$

and illustrates the representation of the actual, extended, generally nonspherical and deformable shape of the particle considered, in this case, a spheroidal ellipsoid.

The following is another example of isounit representing a relativistic system of extended, nonspherical and deformable particles under nonlinear, nonlocal and nonpotential interactions

$$\hat{I}_{Tot} = \Pi_{k=1,2,...,n} Diag.(n_{k1}^2, n_{k2}^2, n_{k3}^2, n_{k4}^2) \times \Gamma(x, v, \psi, \partial\psi, \quad (3.18)...),$$

where Γ is a function characterized by the considered nonlinear, nonlocal and nonpotential interactions, and n_{k4} represents the *density* of the medium in the interior of the particle k with the value $n_{k4} = 1$ for the vacuum, or, equivalently, the local speed of light within physical media $c = c_o/n_{k4}$, in which case n_{k4} is the familiar index of refraction and c_o is the speed of light in vacuum.

Numerous additional examples of isounits exist in the literature [4-11]. Note that the features represented by the isounits are strictly outside any representational capability by the Hamiltonian.

DEFINITION 3.1: Let $F = F(a, +, \times)$ be a field as per Definition 2.1. The *isofields*, first introduced in Ref. (23) of 1978 (see Ref. (12) for a mathematical treatment) are rings $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ whose elements are the *isonumbers*

$$\hat{a} = a \times \hat{I}, \tag{3.19}$$

with associative, distributive and commutative *isosum*

$$\hat{a}\hat{+}\hat{b} = (a + b) \times \hat{I} = \hat{c} \in \hat{F}, \tag{3.20}$$

associative and distributive *isoproduct*

$$\hat{a}\hat{\times}\hat{b} = \hat{a} \times \hat{T} \times \hat{b} = \hat{c} \in \hat{F}, \tag{3.21}$$

additive isounit

$$\hat{0} = 0, \hat{a}\hat{+}\hat{0} = \hat{0}\hat{+}\hat{a} = \hat{a}, \tag{3.22}$$

and*multiplicative isounit*

$$\hat{I} = 1/\hat{T} > 0, \hat{a}\hat{\times}\hat{I} = \hat{I}\hat{\times}\hat{a} = \hat{a}, \forall \hat{a}, \hat{b} \in \hat{F}, \tag{3.23}$$

where $\hat{I}$ is not necessarily an element of F. Isofields are called of the *first (second) kind* when $\hat{I} = 1/\hat{T} > 0$ is (is not) an element of F.

LEMMA 3.1: Isofields of first and second kind are fields (namely, isofields verify all axioms of a field with characteristic zero).

The above property establishes the fact (first identified in Ref. (12) that, by no means, the axioms of a field require that the multiplicative unit be the trivial unit +1, because it can be a negative-definite quantity as for the isodual mathematics, as well as an arbitrary positive-definite quantity, such as a matrix or an integrodifferential operator.

Needless to say, the liftings of the unit and of the product imply a corresponding lifting of all conventional operations of a field. In fact, we have the *isopowers*

$$\hat{a}^{\hat{n}} = \hat{a}\hat{\times}\hat{a}\hat{\times}..., \hat{\times}\hat{a}(ntimes) = a^n \times \hat{I}, \tag{3.24}$$

with particular case

$$\hat{I}^{\hat{n}} = \hat{I}; \tag{3.25}$$

the *isosquare root*

$$\hat{a}^{\hat{1/2}} = a^{1/2} \times \hat{I}^{1/2}; \tag{3.26}$$

the *isoquotient*

$$\hat{a}\hat{/}\hat{b} = (\hat{a}/\hat{b}) \times \hat{I} = (a/b) \times \hat{I}; \tag{3.27}$$

the *isonorm*

$$\hat{|}\hat{a}\hat{|} = |a| \times \hat{I}, \tag{3.28}$$

where $|a|$ is the conventional norm; etc.

Despite their simplicity, the above liftings imply a complete generalization of the conventional number theory particularly for the case of the first kind (in which $\hat{I} \in F$) with implications for all aspects of the theory. As an illustration, the use of the isounit $\hat{I} = 1/3$ implies that "2 multiplied by 3" = 18, while 4 becomes a prime number.

An important contribution has been made by E. Trell (143) who has achieved a proof of Fermat's Last Theorem via the use of isonumbers, thus achieving a proof which is sufficiently simple to be of Fermat's time. A comprehensive study of Santilli's isonumber theory of both first and second kind has been conducted by C.-X. Jiang in monograph (68) with numerous novel; developments and applications. Additional studies on isonumbers have been done by N. Kamiya *et al.* (156) and others (see mathematical papers (10) and proceedings (8)).

The lifting of fields into isofields implies a corresponding lifting of functional analysis into a form known as *isofunctional analysis* studied by J. V. Kadeisvili (132-133), A. K. Aringazin *et al.* (144) and other authors. A review of isofunctional analysis up to 1995 with various developments has been provided by R. M. Santilli in monographs (54,55). We here merely recall the *isofunctions*

$$\hat{f}(\hat{x}) = f(x \times I) \times \hat{I}; \tag{3.29}$$

the *isologarithm*

$$l\hat{o}g_{\hat{e}}a = \hat{I} \times log_e a, l\hat{o}g_{\hat{e}}\hat{e} = \hat{I}, l\hat{o}g_{\hat{e}}\hat{I} = 0; \tag{3.30}$$

the *isoexponentiation*,

$$\hat{e}^{\hat{A}} = \hat{I}\hat{+}\hat{A}\hat{/}\hat{1}!\hat{+}\hat{A}\hat{\times}\hat{A}\hat{/}\hat{2}!\hat{+}... = (e^{\hat{A}\times\hat{T}}) \times \hat{I} = \hat{I} \times (e^{\hat{T}\times\hat{A}}). \tag{3.31}$$

The conventional differential calculus must also be lifted, for consistency, into the *isodifferential calculus* first identified by R. M. Santilli in memoir (14) of 1996, with *isodifferential*

$$\hat{d}\hat{x} = \hat{T} \times d\hat{x} = \hat{T} \times d(x \times \hat{I}), \tag{3.32}$$

which, for the case when $\hat{I}$ does not depend on x, reduces to

$$\hat{d}\hat{x} = dx; \tag{3.33}$$

the *isoderivatives*

$$\hat{\partial}\hat{f}(\hat{x})/\hat{\partial}\hat{x} = \hat{I} \times [\partial f(\hat{x})/\partial\hat{x}], \tag{3.34}$$

and other similar properties. The indicated invariance of the differential under isotopy, $\hat{d}\hat{x} = dx$, illustrates the reason why the isodifferential calculus has remained undetected since Newton's and Leibnitz's times.

3.4: Isotopologies, isospaces and isogeometries. Particularly important for these notes is the *isotopy of the Euclidean topology* independently identified by G. T. Tsagas and D. S. Sourlas (139) and R. M. Santilli (14), as well as the *isotopies of the Euclidean,* *Minkowskian, Riemannian and symplectic geometries,* first identified by Santilli in various works (see Refs. (14,15,29,54,55) and references quoted therein). We cannot possibly review here these advances for brevity.

We merely mention that any given n-dimensional metric or pseudometric space $S(x, m, R)$ with basic unit $I = Diag.(1, 1, ..., 1)$, local coordinates $x = (x^i), i = 1, 2, .., n$, $n \times n$-dimensional metric m and invariant $x^2 = x^i \times m_{ij} \times x^j \in R$ is lifted into the *isospaces* $\hat{S}(\hat{x}, \hat{m}, \hat{R})$ with *isocoordinates, isometric and isoinvariant* respectively given by

$$I = Diag.(1, 1, ..., 1) \rightarrow \hat{I}_{n \times n}(x, v, ...) = 1/\hat{T}(x, v, ...), \qquad (3.35a)$$

$$x \rightarrow \hat{x} = x \times \hat{I}, m \rightarrow \hat{m} = \hat{T}(x, v, ...)\ m, \qquad (3.35b)$$

$$x^2 = x^i \times m_{ij} \times x^j \times I \in R \rightarrow \hat{x}^{\hat{2}} = \hat{x}^i \hat{\times} \hat{m}_{ij} \hat{\times} \hat{x}^j \times \hat{I} =$$

$$= \{x^i \times [\hat{T}(x, v, ...) \times m] \times x^j\} \times \hat{I} \in \hat{R}. \qquad (3.35c)$$

where one should note that $\hat{m}$ is an *isomatrix,* namely, a matrix whose elements are isonumbers (thus being multiplied by $\hat{I}$ to be in $\hat{R}$) and all operation are isotopic (in this way the calculation of the value of an *isodeterminant* cancels out all multiplications by $\hat{I}$ except the last, thus correctly producing an isonumber).

An inspection of the functional dependence of the isometric $\hat{m} = \hat{T}(x, v, ...) \times m$ then reveals that *isospaces* $\hat{S}(\hat{x}, \hat{m}\hat{R})$ *unify all possible spaces with the same dimension and signature.* As an illustration, the isotopy of the 3-dimensional Euyccldiean space includes as particular case the 3-dimensional Riemannian, Finslerian as well as any other space with the same dimension and signature $(+, +, +)$ (in view of the positive-definiteness of $\hat{I}$. Broader unifications are possible in the event such positive-definiteness is relaxed.

Since the isotopies preserve the original axioms, the unification of the Euclidean and Riemannian geometry implies the reduction of *Riemannian* geometry to *Euclidean* axioms on isospaces over isofields. In turn, such a geometric unification has far reaching implications, e.g., for relativities, grand unifications and cosmologies (see later on).

It should be mentioned that "deformations" of conventional geometries are rather fashionable these days in the physical and mathematical literature. However, these deformations are generally afflicted by the catastrophic inconsistencies of Theorem 3.1 because, when the original geometry is canonical, the deformed geometry is noncanonical, thus losing the invariance needed for consistent applications. The isotopies of conventional geometries were constructed precisely to avoid such inconsistencies by reconstructing invariance on isospaces over isofield while having a fully noncanonical structure, as shown below.

Therefore, for "deformations" the generalized metric $\hat{m} = \hat{T} \times m$ and related invariant are referred to conventional units and fields R, while for "isotopies" the same generalized metric $\hat{m} = \hat{T} \times m$ is referred to a isounit which is the *inverse* of the deformation of the metric, $\hat{I} = \hat{T}^{-1}$. While the deformed geometry verify axioms different then the original ones, the lifting of the original metric m by the matrix $\hat{T}$ while the basic unit is lifted by the *inverse* amount implies the preservation of the original axioms, with consequential unifications of different geometries.

Particularly intriguing are the isotopies of the symplectic geometry, known as *isosymplectic geometry* (14) which are based on the following *fundamental isosymplectic two-isoform*

$$\hat{d}\hat{p}\hat{\wedge}\hat{d}\hat{r} = \hat{\omega} = dp \wedge dr = \omega, \qquad (3.36)$$

due to the fact that, for certain geometric reasons, the isounit of the variable p in the cotangent bundle (phase space) is the inverse that of x (i.e., when $\hat{I} = 1/\hat{T}$ is the isounit for x, that for p is $\hat{T} = 1/\hat{I}$). The invariance $\hat{\omega} = \omega$ provide a reason why the isotopies of the symplectic geometry have escaped identification by mathematicians for over one century.

Despite their simplicity, the isotopies of the symplectic geometry have vast implications, e.g., a broader quantization leading to a structural generalization of quantum mechanics known as *hadronic mechanics,* as outlined below.

3.5: Lie-Santilli isotheory and its isodual. As well known, Lie's theory (4) is based on the conventional (left and right) unit $I = Diag.(1, 1, ..., 1)$ of the universal enveloping associative algebra. The lifting $I \rightarrow \hat{I}(x, ...)$ implies the lifting of the entire Lie theory, first proposed by R. M. Santilli in Ref. (23) of 1978 and then studied in numerous works (see, e.g., memoir (14) and monographs (51,54,55)). The isotopies of Lie's theory are today known as the *Lie-Santilli isotheory* following studies by numerous mathematicians and physicists (see the monographs by D. S. Sourlas and Gr. Tsagas (64), J. V. Kadeisvili (66), R. M. Falcon Ganfornina and J. Nunez Valdez (67), proceedings [8] and contributions quoted therein).

Let $\xi(L)$ be the universal enveloping associative algebra of an N-dimensional Lie algebra L with (Hermitean) generators $X = (X_i)$, i = 1, 2, ..., n, and corresponding Lie transformation group G over the reals R. The Lie-Santilli isotheory is characterized by:

(I) The *universal enveloping isoassociative algebra* $\hat{\xi}$ with infinite-dimensional basis characterizing the *Poincaré-Birkhoff-Witt-Santilli isotheorem*

$$\hat{\xi} : \hat{I}, \hat{X}_i, \hat{X}_i \hat{\times} \hat{X}_j, \hat{X}_i \hat{\times} \hat{X}_j \hat{\times} \hat{X}_k, ..., i \leq j \leq k; \qquad (3.37)$$

where the "hat" on the generators denotes their formulation on isospaces over isofields;

(II) The *Lie-Santilli isoalgebras*

$$\hat{L} \approx (\hat{\xi})^- : [\hat{X}_i \hat{,} \hat{X}_j] = \hat{X}_i \hat{\times} \hat{X}_j - \hat{X}_j \hat{\times} \hat{X}_i = \hat{C}_{ij}^k \hat{\times} \hat{X}_k; \qquad (3.38)$$

(III) The *Lie-Santilli isotransformation groups*

$$\hat{G} : \hat{A}(\hat{w}) = (\hat{e}^{\hat{i} \hat{\times} \hat{X} \hat{\times} \hat{w}}) \hat{\times} \hat{A}(\hat{0}) \hat{\times} (\hat{e}^{-\hat{i} \hat{\times} \hat{w} \hat{\times} \hat{X}}) = (e^{i \times \hat{X} \times \hat{T} \times w}) \times A(0) \times (e^{-i \times w \times \hat{T} \times \hat{X}}), \qquad (3.39)$$

where $\hat{w} \in \hat{R}$ are the *isoparameters;* the *isorepresentation theory;* etc.

The non-triviality of the above liftings is expressed by the appearance of the isotopic element $\hat{T}(x, ...)$ at all levels (I), (II) and (III) of the isotheory. The arbitrary functional dependence of $\hat{T}(x, ...)$ then implies the achievement of the desired main features of the isotheory which can be expressed by the following:

LEMMA 3.2 (14): Lie-Santilli isoalgebras on conventional spaces over conventional fields are generally nonlocal, nonlinear and noncanonical, but they verify locality, linearity and canonicity when formulated on isospaces over isofields.

To illustrate the Lie-Santilli isotheory in the operator case, consider the eigenvalue equation on $\mathcal{H}$ over C, $H(x, p, \psi, ...) \times |\psi >= E \times |\psi >$. This equation is nonlinear in the wavefunction, thus violating the superposition principle and preventing the study of composite nonlinear systems, as indicated earlier. However, under the factorization

$$H(x, p, \psi, ...) = H'(x, p) \times \hat{T}(x, p, \psi, ...), \tag{3.40}$$

the above equation can be reformulated *identically* in the isotopic form

$$H(x, p, \psi, ...) \times |\psi >= H'(x, p) \times \hat{T}(x, p, \psi, ...) \times |\psi >= H' \hat{\times} |\psi >= E \times |\psi >= \hat{E} \hat{\times} |\psi >, \tag{3.41}$$

whose reconstruction of linearity on isospaces over isofields (called *isolinearity* (14)) is evident and so is the verification of the isosuperposition principle with resulting applicability of isolinear theories for the study of composite nonlinear systems. Similar results occur for the reconstruction on isospace over isofields of locality (called *isolocality*) and canonicity (called *isocanonicity*).

A main role of the isotheory is then expressed by the following property:

LEMMA 3.3 (29): Under the condition that $\hat{I}$ is positive-definite, isotopic algebras and groups are locally isomorphic to the conventional algebras and groups, respectively.

Stated in different terms, the Lie-Santilli isotheory *was not* constructed to characterize new Lie algebras, because all Lie algebras over a field of characteristic zero are known. On the contrary, the Lie-Santilli isotheory has been built to characterize *new realizations* of known Lie algebras generally of nonlinear, nonlocal and noncanonical character as needed for a deeper representation of valence bonds or, more generally, systems with nonlinear, nonlocal and noncanonical interactions.

The mathematical implications of the Lie-Santilli isotheory are significant. For instance, Gr. Tsagas (142) has shown that *all simple non-exceptional Lie algebras of dimension N can be unified into one single Lie-Santilli isotope of the same dimension,* while studies for the inclusion of exceptional algebras in this grand unification of Lie theory are under way In fact, the characterization of different simple Lie algebras, including the transition from compact to noncompact Lie algebras, can be characterized by *different realizations of the isounit while using a unique form of generators and of structure constants* (see the first examples for the SO(3) algebra in Ref. (23) of 1978 and numerous others in the quoted literature).

The physical implications of the Lie-Santilli isotheory are equally significant. We here mention the *reconstruction as exact at the isotopic level of Lie symmetries when believed to be broken under conventional treatment.* In fact, R. M. Santilli has proved: the exact reconstruction of the rotational symmetry for all ellipsoidical deformations of the sphere (12); the exact SU(2)-isospin symmetry under electromagnetic interactions (28,33); the exact Lorentz symmetry under all (sufficiently smooth) signature-preserving deformations of the Minkowski metric (26); and the exact reconstruction of parity under weak interactions (55). R. Mignani (180) has studied the exact reconstruction of the SU(3) symmetry under various symmetry-breaking terms. In all these cases the reconstruction of the exact symmetry has been achieved by merely embedding all symmetry breaking terms in the isounit.

The construction of the isodual Lie-Santilli isotheory for antimatter is an instructive exercise for interested readers.

The main physical theories characterized by isomathematics are given by:

3.6: Iso-Newtonian Mechanics and its isodual. As it is well known, Newton (1) had to construct the differential calculus as a necessary pre-requisite for the formulation of his celebrated equations. Today we know that Newton's equations can only represent *point-particles* due to the strictly local-differential character of the underlying Euclidean topology. The fundamental character of Newtonian Mechanics for all scientific inquiries is due to the preservation at all subsequent levels of study (such as Hamiltonian mechanics, quantum mechanics, quantum chemistry, quantum field theory, etc.) of:

1) The underlying Euclidean topology;
2) The differential calculus; and
3) The notion of point particle.

By keeping in mind Newton's teaching, the author has dedicated primary efforts to the isotopic lifting of the conventional differential calculus, topology and geometries (14) as a necessary pre-requisite for a structural generalization of Newton's equations into a form representing *extended, nonspherical and deformable particles* under action-at-a-distance/potential as well as contact/nonpotential forces.

The need for such a lifting is due to the fact that *point particles cannot experience contact-resistive forces.* This feature has lead to subsequent theories, such as Hamiltonian and quantum mechanics, which solely admit action-at-a-distance/potential forces among point particles. Such a restriction is indeed valid for a number of systems, such as planetary systems at the classical level and atomic systems at the operator level, because the large distances among the constituents permit an effective point–like approximation of particles.

However, when interactions occur at short distances, as in the case of electron valence bonds (Figure 2) or the mutual penetration of the wavepackets of particles in general, the point-like approximation is no longer sufficient and a representation of the actual, extended, generally nonspherical and deformable shape of particles is a necessary pre-requisite to admit contact nonpotential interactions.

By recalling the fundamental character of Newtonian mechanics for all of sciences, the achievement of a consistent representation of the contact interactions of valence electron bonds at the *operator* level requires the prior achievement of a consistent *Newtonian* representation.

To outline the needed isotopies, let us recall that Newtonian mechanics is formulated on the Kronecker product $S_{tot} = S_t \times S_x \times S_v$ of the one dimensional space S_t representing time t, the tree dimensional Euclidean space S_x of the coordinates $x = (x^k_\alpha)$ (where $k = 1, 2, 3$ are the Euclidean axes and $\alpha = 1, 2, ..., n$ represents the number of particles), and the velocity space $S_v, v = dx/dt$.

It is generally assumed that all variables t, x, and v are defined on the same field of real numbers R. However, the unit of time is the *scalar* $I = 1$, while the unit of the Euclidean space is the *matrix* $I = Diag.(1, 1, 1)$. Therefore, on rigorous grounds, the representation space of Newtonian mechanics $S_{tot} = S_1 \times S_x \times S_v$ must be defined on the Kronecker product of the corresponding fields $R_{tot} = R_t \times R_x \times R_v$ with total unit $I_{Tot} = 1 \times Diag.(1, 1, 1)_x \times Diag.(1, 1, 1)_v$.

Newtonian systems requested for the isotopies are given by the so-called *closed-isolated non-Hamiltonian systems* (51), namely, systems which are closed-isolated from the rest of the universe, thus verifying all ten Galilean total conservation laws, yet they admit internal non-Hamiltonian forces due to contact interactions.

A typical illustration is given by the structure of Jupiter which, when considered as isolated from the rest of the universe, does indeed verify all Galilean conservation laws, yet its internal structure is clearly non-Hamiltonian due to vortices with *varying angular momentum* and similar internal dissipative effects. In essence, contact nonpotential forces produce internal exchanges of energy, linear and angular momentum but always in such a manner to verify total conservation laws.

A Newtonian representation of closed-isolated non-Hamiltonian systems of extended particles is given by(Ref. (51), page 236)

$$m_\alpha \times a_{k\alpha} = m_\alpha \times \frac{dv_{k\alpha}}{dt} = F_\alpha(t, x, v) = F^{SA}_\alpha(x) + F^{NSA}_\alpha(t, x, v), \qquad (3.42a)$$

$$\sum_{\alpha=1,...,n} F^{NSA}_\alpha = 0, \qquad (3.42b)$$

$$\sum_{\alpha=1,...,n} \mathbf{x}_\alpha \bigodot \mathbf{F}^{NSA}_\alpha = 0, \qquad (3.42c)$$

$$\sum_{\alpha=1,...,n} \mathbf{x}_\alpha \bigwedge \mathbf{F}^{NSA}_\alpha = 0, \qquad (3.42d)$$

where: SA (NSA) stands for *variational selfadjointness (variational nonselfadjointness),* namely, the verification (violation) of the integrability conditions for the existence of a potential, and conditions (3.9b), (3.9c) and (3.9d) assure the verification of all ten Galilean conservation laws (for the total energy, linear momentum, angular momentum, and uniform motion of the center of mass). The restrictions to F^{NSA} verifying the above conditions is tacitly assumed hereon.

The isotopies of Newtonian mechanics, also called *Newton-Santilli isomechanics* (63-68), requires the use of the *isotime* $\hat{t} = t \times \hat{I}_t$ with isounit $\hat{I}_t = 1/\hat{T}_t$ and related isofield $\hat{R}_t$, the *isocoordinates* $\hat{x} = (\hat{x}^k_\alpha) = x \times \hat{I}_x$, with isounit $\hat{I}_x = 1/\hat{T}_x$ and related isofield $\hat{R}_x$, and the *isospeeds* $\hat{v} = (v_{k\alpha}) = v \times \hat{I}_v$ with isounit $\hat{I}_v = 1/\hat{T}_v$ and related isofield $\hat{R}_v$.

IsoNewtonian Mechanics is then formulated on the Kronecker product of isospaces $\hat{S}_{Tot} = \hat{S}_t \times \hat{S}_x \times \hat{S}_v$ over the Kronecker product of isofields $\hat{R}_t \times \hat{R}_x \times \hat{R}_v$. The isospeed is the given by

$$\hat{v} = \frac{\hat{d}\hat{x}}{\hat{d}\hat{t}} = \hat{I}_t \times \frac{d(x \times \hat{I}_x)}{dt} = v \times \hat{I}_t \times \hat{I}_x + x \times \hat{I}_t \times \frac{d\hat{I}_x}{dt} = v \times \hat{I}_v, \tag{3.43a}$$

$$\hat{I}_v = \hat{I}_t \times \hat{I}_x \times (1 + x \times \hat{T}_x \times \frac{d\hat{I}_x}{dt}. \tag{3.43b}$$

The *Newton-Santilli isoequation and its isodual,* first proposed in memoir (14) of 1996 (where the isodifferential calculus was first achieved) can be written

$$\hat{m}_\alpha \hat{\times} \frac{\hat{d}\hat{v}_{k\alpha}}{\hat{d}\hat{t}} = -\frac{\hat{\partial}\hat{V}(\hat{x})}{\hat{\partial}\hat{x}^k_\alpha}. \tag{3.44}$$

namely, *the equations are conceived in such a way to formally coincide with the conventional equations for selfadjoint forces, $F^{SA} = -\partial V/\partial x$, while all nonpotential forces are represented by the isounits or, equivalently, by the isodifferential calculus.* Such a conception is the only one known which permits the representation of extended particles with contact interactions which is invariant (thus avoiding the catastrophic inconsistencies of Theorem 3.1) and achieves closure, namely, the verification of all ten Galilean conservation laws.

An inspection of Eqs. (3.10) is sufficient to see that *iso-Newtonian mechanics reconstructs canonicity on isospace over isofields,* thus avoiding Theorem 3.1. Note that this would not be the case if nonselfadjoint forces appear in the right hand side of Eqs. (3.10) as in Eqs. (3.9a).

The verification of all Galilean conservation laws is equally established by a visual inspection of Eqs. (3.10) since their symmetry, the *iso-Galilean symmetry* with structure (3.8), is the Galilean symmetry, only formulated on isospace over isofields (53). By recalling that conservation laws are represented by the generators of the underlying symmetry, conventional total conservation laws then follow from the fact that *the generator of the conventional Galilean symmetry and its isotopic lifting coincide.*

When projected in the conventional Newtonian space S_{Tot}, Eqs. (3.10) can be explicitly written

$$\hat{m} \hat{\times} \frac{\hat{d}\hat{v}}{\hat{d}\hat{t}} = m \times \hat{I}_t \times \frac{d(v \times \hat{I}_v)}{dt} =$$

$$= m \times a \times \hat{I}_t \times \hat{I}_v + m \times v \times \hat{I}_t \times \frac{d\hat{I}_v}{dt} = -\frac{\hat{\partial}\hat{V}(\hat{x})}{\hat{\partial}\hat{x}} = -\hat{I}_x \times \frac{\partial V}{\partial x}, \tag{3.46}$$

that is

$$m \times a = -\hat{T}_t \times \hat{T}_v \times \hat{I}_x \times \frac{\partial V}{\partial x} - m \times v \times \hat{T}_v \times \frac{d\hat{I}_v}{dt}, \tag{3.47}$$

with necessary and sufficient conditions for the representation of all possible SA and NSA forces

$$\hat{I}_t \times \hat{I}_v \times \hat{I}_x = I, \hat{I}_x = 1/\hat{T}_t \times \hat{T}_x, \tag{3.48a}$$

$$m \times v \times \hat{T}_v \times \frac{d\hat{I}_v}{dt} = F^{NSA}(t, x, v), \qquad (3.48b)$$

which always admit a solution, since they constitute a system of 6n *algebraic* (rather than differential) equations in the $6n+1$ unknowns given by $\hat{I}_t$, and the diagonal $\hat{I}_x$ and $\hat{I}_v$.

As an illustration, we have the following equations of motion of an *extended* particle with the ellipsoidal shape experiencing a resistive force $F^{NSA} = -\gamma \times v$ because moving within a physical medium

$$m \times a = -\gamma \times v \qquad (3.49a)$$

$$\hat{I}_v = Diag.(n_1^2, n_2^2, n_3^2) \times e^{\gamma \times t/m}. \qquad (3.49b)$$

Interested readers can then construct the representation of *any* desired NSA forces (see also memoir (14) for other examples).

Note the natural appearance of the velocity dependence, as typical of resistive forces. Note also that the representation of the extended character of particles occurs only in isospace because, when Eqs. (3.10) are projected in the conventional Newtonian space, all isounits cancel out and the point characterization of particles is recovered. Note finally the *direct universality* of the Newton-Santilli isoequations, namely, their capability of representing all infinitely possible Newton's equations in the frame of the observer.

As indicated earlier, Eqs. (3.42) can only describe a system of *particles.* The construction of the *isodual Newton-Santilli isoequations* for the treatment of a system of *antiparticles* is left to the interested reader.

We finally indicate that the invariance of closed non-Hamiltonian systems (3.42) is given by the *Galilei-Santilli isosymmetry* $\hat{G}(3.1)$ and their isoduals by $\hat{G}^d(3.1)$ (see Refs. (52,53) for brevity).

3.7: Iso-Hamiltonian Mechanics and its isodual. Eqs. (3.10) admit the analytic representation in terms of the following *isoaction principle* (14)

$$\hat{\delta}\hat{\mathcal{A}}(\hat{t}, \hat{x}) = \hat{\delta}\hat{\int}(\hat{p}_{k\alpha}\hat{\times}_p d\hat{x}^k_\alpha) - \hat{H}\hat{\times}_t d\hat{t} =$$

$$= \delta\hat{\int}[p_{k\alpha} \times \hat{T}_x(t, x, p, ...) \times d(x^k_\alpha \times \hat{I}_x) - H \times \hat{T}_t(t, x, p, ...)d(t \times \hat{I}_t) = 0. \qquad (3.50)$$

Note the main result permitted by the isodifferential calculus, consisting in the *reduction of an action functional of arbitrary power in the linear momentum (arbitrary order) to that of first power in p first order.* Since the optimal control theory and the calculus of variation depend on the first order character of the action functional, the above reduction has important implications, such as the treatment of extended objects moving within resistive media apparently for the first time via the optimal control theory, since a first order conventional action is impossible for the systems considered..

Note that when the isounits are constant, isoaction the isoaction coincides with the conventional action. This illustrates the apparent reason why the isotopies of the action principle creeped in un-noticed for over one century.

It is easy to prove that the above isoaction principle characterizes the *Hamilton-Santilli isoequations* (14)

$$\frac{\hat{d}\hat{x}}{\hat{d}\hat{t}} = \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{p}} = \frac{\hat{p}}{\hat{m}} = \frac{\hat{p}}{\hat{m}}, \frac{\hat{d}\hat{p}}{\hat{d}\hat{t}} = -\frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{x}} = \hat{F}^{SA} + \hat{F}^{NSA}, , \qquad (3.51a),$$

$$\hat{H} = \sum_{\alpha=1,...,n} \frac{\hat{p}_{k\alpha} \hat{\times}_p \hat{p}^k_\alpha}{\hat{2} \hat{\times} \hat{m}_\alpha} \qquad (3.51b)$$

$$\hat{I}_t = 1, \hat{I}_x = I + F^{NSA}/F^{SA}, \hat{I}_p = \hat{T}_x, \qquad (3.51c)$$

where one should note the real-valued, symmetric and positive-definite character of all isounits, and corresponding *Hamilton-Jacobi-Santilli isoequations*

$$\frac{\hat{\partial}\hat{\mathcal{A}}}{\hat{\partial}\hat{t}} + \hat{H} = 0, \frac{\hat{\partial}\hat{\mathcal{A}}}{\hat{\partial}\hat{x}^k_\alpha} - \hat{p}_{k\alpha} = 0. \qquad (3.52)$$

As it was the case for Eqs. (3.10), iso-Hamiltonian mechanics has been conceived to coincide at the abstract level with the conventional formulation. Nevertheless, the following main differences occur:

1) Hamiltonian mechanics can only represent point particles while its isotopic covering can represent the actual, extended, nonspherical and deformable shape of particles via the simply identification of isounits (3.11c);

2) Hamiltonian mechanics can only represent a rather restricted class of Newtonian systems, those with potential forces, while its isotopic covering is directly universal for all possible (sufficiently smooth) SA and NSA Newtonian systems;

3) All NSA forces are represented by the isounits or, equivalently, by the isodifferential calculus, thus permitting their *invariant* description, since iso-Hamiltonian mechanics clearly reconstructs canonicity on isospaces over isofields.

Iso-Hamiltonian mechanics as outline above can only described closed non-Hamiltonian systems of *particles.* The construction of its isodual for *antiparticles* is an instructive exercise for interested readers.

3.8: Isotopic Branch of nonrelativistic Hadronic Mechanics and its isodual. The preservation of the form of Newton's and Hamilton's equations has far reaching implications, since it permit a simple lifting of quantization, resulting in a generalization of quantum mechanics known under the name of *isotopic branch of Hadronic Mechanics* (see memoir (31) for a general review), which permits, apparently for the first time, an axiomatically consistent and invariant representation of extended particles under linear and nonlinear, local and nonlocal, and potential as well as nonpotential interactions.

Recall that the conventional naive or symplectic quantization $\mathcal{A} \to -i \times \hbar \times Ln\psi$ is solely applicable for first-order action functionals $\mathcal{A}(t, x)$ and, as such, it is not applicable to the isoaction $\hat{\mathcal{A}}(\hat{t}, \hat{x}) = \hat{\mathcal{A}}(t, x, p, ...)$ due to its higher order when formulated on conventional spaces. Nevertheless, it is easy to show that the following *naive isoquantization* holds (for $\hat{\hbar}$ replaced by $\hat{I}_x$)

$$\hat{\mathcal{A}}(\hat{t},\hat{x}) \rightarrow -\hat{i}\hat{\times}\hat{T}_x\hat{\times}\hat{L}n\hat{\psi}(\hat{t},\hat{x}), \tag{3.53}$$

which, when applied to Eqs. (3.12) permits the map here expressed for the case when $\hat{I}_x$ is a constant (see Ref. (55) for the general case)

$$\frac{\hat{\partial}\hat{\mathcal{A}}}{\hat{\partial}\hat{t}} + \hat{H} = 0 \rightarrow -i \times \hat{I}_t \times \frac{\partial\hat{\psi}}{\partial t} + \hat{H} \times \hat{T}_x \times \hat{\psi} = 0, \tag{3.54a}$$

$$\frac{\hat{\partial}\hat{\mathcal{A}}}{\hat{\partial}\hat{x}^k_\alpha} - \hat{p}_{k\alpha} = 0 \rightarrow -i \times \hat{I}_x \times \frac{\partial\hat{\psi}}{\partial x} - \hat{p} \times \hat{T}_x \times \hat{\psi} = 0. \tag{3.54b}$$

The above equations can be more properly formulated over the *iso-Hilbert space* (25) $\hat{\mathcal{H}}$ with *isostates* $|\hat{\psi}(\hat{t},\hat{x}) >$ and *isoinner product* $< \hat{\psi}|\hat{\times}|\hat{\psi} > \times\hat{I}$ over the isofield $\hat{C}$ (see memoir (31) for a review). The new mechanics is characterized by the *iso-Schroedinger equations* (first derived in Refs. (25,179) with ordinary mathematics and first formulated via the isodifferential calculus in Ref. (14))

$$\hat{i}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{t}}|\hat{\psi} >= \hat{H}\hat{\times}|\hat{\psi} >= \hat{H}(\hat{x},\hat{p})\times\hat{T}(\hat{x},\hat{p},\hat{\psi},\hat{\partial}\hat{\psi},....)\times|\hat{\psi} >= \hat{E}\hat{\times}|\hat{\psi} >= E\times|\hat{\psi} >, \tag{3.55a}$$

$$\hat{p}_k\hat{\times}|\hat{\psi} >= -\hat{i}\hat{\times}\hat{\partial}_k|\hat{\psi} >= -i \times \hat{I}^i_k \times \partial_i|\hat{\psi} >, \hat{I}\hat{\times}|\hat{\psi} >= |\hat{\psi} >, \tag{3.55b}$$

and the *iso-Heisenberg equations* (first derived in Ref. (38) via conventional mathematics and first formulated via the isodifferential calculus in Ref. (14))

$$\hat{i}\hat{\times}\frac{\hat{d}\hat{A}}{\hat{d}\hat{t}} = [\hat{A}\hat{,}\hat{H}] = \hat{A}\hat{\times}\hat{H}\hat{-}\hat{H}\hat{\times}\hat{A} =$$

$$= \hat{A} \times \hat{T}(\hat{t},\hat{x},\hat{p},\hat{\psi},\hat{\partial}\hat{\psi},...) \times \hat{H} - \hat{H} \times \hat{T}(\hat{t},\hat{x},\hat{p},\hat{\psi},\hat{\partial}\hat{\psi},...) \times \hat{A}, \tag{3.56a}$$

$$[\hat{x}^i\hat{,}\hat{p}_j] = \hat{i}\hat{\times}\hat{\delta}^i_j = i \times \delta^i_j \times \hat{I}, [\hat{x}^i, \hat{x}^j] = [\hat{p}_i, \hat{p}_j] = 0. \tag{3.56b}$$

A first important property the reader can easily prove is that *iso-Hermiticity coincides with conventional Hermiticity.* Consequently, *all quantities which are observable for quantum mechanics remain observable under isotopies.* In particular, it is equally easy to prove that *all Hermitean quantities which are conserved for quantum mechanics remain conserve under isotopies,* again, because the symmetries of Schroedinger' and iso-Schroedinger's equations are isomorphic and their generators coincide.

The above results implies the existence of a *new notion of bound state of particles* as the operator image of closed non-Hamiltonian systems (3.9), namely, a bound state admitting internal Hamiltonian as well as nonlinear, nonlocal and nonpotential interactions while preserving conventional total conservation laws. Note that these are precisely the characteristics needed for quantitative studies of electron valence bonds, as well as, more generally, bound states of particles at shot mutual distances.

Another important property of hadronic mechanics is that, in view of the lack of general commutativity between $\hat{H}$ and $\hat{T}$, the *iso-Schroedinger and iso-Heisenberg's equations have a nonunitary time evolution* when formulated on conventional Hilbert spaces over conventional fields,

$$|\hat{\psi}(t)>=(e^{i\times H\times\hat{T}\times t})\times|\hat{\psi}(0)>=U(t)\times|\hat{\psi}(0)>, U\times U^{\dagger}\neq I, \tag{3.57}$$

However, *all nonunitary transforms admit an identical reformulation as isounitary transform on iso-Hilbert spaces,*

$$U\times U^{\dagger}\neq I, U=\hat{U}\times\hat{T}^{1/2}, \tag{3.58a}$$

$$\hat{U}\hat{\times}\hat{U}^{\dagger}=\hat{U}^{\dagger}\hat{\times}\hat{U}=\hat{I}. \tag{3.58b}$$

The above property is a *necessary condition* to exit from the class of equivalence of quantum mechanics, thus illustrating the nontriviality of the lifting.

Yet another property is that nonlinear Schroedinger's equations cannot represent composite systems because of the violation of the superposition principle, while hadronic mechanics resolves this limitation. In fact,, all nonlinear Schroedinger' s equations can be *identically* rewritten in the isotopic form with the embedding of all nonlinear terms in the isotopic element,

$$H(x,p,\psi)|\psi>=H'(x,p)\times\hat{T}(x,p,\psi),...)\times|\psi>=E\times|\psi>, \tag{3.59}$$

under which linearity, and, therefore, the superposition principle, are trivially reconstructed in isospace over isofields.

The *isoexpectation values* of an observable $\hat{A}$ on $\hat{\mathcal{H}}$ over $\hat{C}$ are given by

$$\frac{<\hat{\psi}|\hat{\times}\hat{A}\hat{\times}|\hat{\psi}>}{<\hat{\psi}|\hat{\times}|\hat{\psi}>}\times\hat{I}\in\hat{C}. \tag{3.60}$$

It is easy to prove that the isoexpectation values coincide with the isoeigenvalues, as in the conventional case. In particular, *the isoexpectation value of the isounit recovers Planck's unit*

$$\frac{<\hat{\psi}|\hat{\times}\hat{I}\hat{\times}|\hat{\psi}>}{<\hat{\psi}|\hat{\times}|\hat{\psi}>}=\hbar=1. \tag{3.61}$$

Note also the following *invariance of Hilbert's inner product under isotopy* (for the case when the isotopic element does not depend on the integration variable)

$$<\psi|\times|\psi>\times I\equiv<\psi|\times\hat{T}\times|\psi>\times\hat{I}, \tag{3.62}$$

which invariance explains why the isotopies of Hilbert spaces remained un-discovered since Hilbert's time even though they have the important implication of causing a structural generalization of the conventional formulation of quantum mechanics. Note that, despite its simplicity, invariance (3.62) required the prior identification of *new numbers,* those with arbitrary unit $\hat{I}$.

The latter properties establish that *the isotopic branch of hadronic mechanics coincides with quantum mechanics at the abstract, realization-free level.* This feature is important to assure the axiomatic consistency of hadronic mechanics, as well as to clarify the fact that *hadronic mechanics is not a new theory, but merely a novel realization of the abstract axioms of quantum mechanics.*

The *isodual isotopic branch of hadronic mechanics* for the treatment of antimatter is given by the image of the theory under map (2.4) and its outline is here omitted for brevity.

For additional intriguing features of hadronic mechanics, interested readers can inspect memoir (31) and monograph (55).

3.9: Invariance of isotopic theories. The invariance of the basic axioms and numerical predictions pf isotopic theories under time as well as other transforms was first achieved in Refs. (14,31). It can be proved on isospaces over isofields by reformulating any given, nonunitary transform in the *isounitary form,*

$$W \times W^{\dagger} = \hat{I}, W = \hat{W} \times \hat{T}^{1/2}, W \times W^{\dagger} = \hat{W} \hat{\times} \hat{W}^{\dagger} = \hat{W}^{\dagger} \hat{\times} \hat{W} = \hat{I}, \tag{3.63}$$

and then showing that the basic isoaxioms are indeed invariant, i.e.,

$$\hat{I} \rightarrow \hat{I}' = \hat{W} \hat{\times} \hat{I} \hat{\times} \hat{W}^{\dagger} = \hat{I}, \tag{3.64a}$$

$$\hat{A} \hat{\times} \hat{B} \rightarrow \hat{W} \hat{\times} (\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{W}^{\dagger} =$$
$$= (\hat{W} \times \hat{T} \times A \times \hat{T} \times \hat{W}^{\dagger}) \times (\hat{T} \times \hat{W}^{\dagger})^{-1} \times \hat{T} \times (\hat{W} \times \hat{T})^{-1} \times (\hat{W} \times \hat{T} \times \hat{B} \times \hat{T} \times \hat{W}^{\dagger}) =$$
$$= \hat{A}' \times (\hat{W}^{\dagger} \times \hat{T} \times \hat{W})^{-1} \times \hat{B}' = \hat{A}' \times \hat{T} \times \hat{B}' = \hat{A}' \hat{\times} \hat{B}', etc. \tag{3.64b}$$

The invariance is ensured by the *numerically invariant values of the isounit $\hat{I}$ and of the isotopic element $\hat{T}$ under nonunitary-isounitary transforms*, namely,

$$\hat{I} \rightarrow \hat{I}' = \hat{I}, \hat{T} \rightarrow \hat{T}' = \hat{T}, \hat{\times} \rightarrow \hat{\times}' = \hat{\times}. \tag{3.65}$$

The resolution of the catastrophic inconsistencies of Theorem 3.1 is then consequential.

The achievement of invariant for classical noncanonical formulations is equivalent to the preceding nonunitary one and its explicit form is left to the interested reader for brevity.

3.10: Simple construction of isotheories. A simple method has been identified in Refs. (14,31) for the construction of the entire isomathematics and its physical applications. It consists in:

(i) representing all conventional interactions with a Hamiltonian H and all nonhamiltonian interactions and effects with the isounit $\hat{I}$;

(ii) identifying the latter interactions with a nonunitary transform

$$U \times U^{\dagger} = \hat{I} \neq I; \tag{3.66}$$

and

(iii) subjecting the totality of conventional mathematical and physical quantities and all their operations to said nonunitary transform,

$$I \rightarrow \hat{I} = U \times I \times U^\dagger = 1/\hat{T}, a \rightarrow \hat{a} = U \times a \times U^\dagger = a \times \hat{I}, \tag{3.67a}$$

$$a \times b \rightarrow U \times (a \times b) \times U^\dagger = (U \times a \times U^\dagger) \times (U \times U^\dagger)^{-1} \times (U \times b \times U^\dagger) = \hat{a} \hat{\times} \hat{b}, \tag{3.67b}$$

$$e^A \rightarrow U \times e^A \times U^\dagger = \hat{I} \times e^{\hat{T} \times \hat{A}} = (e^{\hat{A} \times \hat{T}}) \times \hat{I}, \tag{3.67c}$$

$$[X_i, X_j] \rightarrow U \times [X_i X_j] \times U^\dagger = [\hat{X}_i \hat{,} \hat{X}_j] = U \times (C_{oj}^k \times X_k) \times U^\dagger = \hat{C}_{ij}^k \hat{\times} \hat{X}_k =$$
$$= C_{ij}^k \times \hat{X}_k, \tag{3.67d}$$

$$< \psi | \times | \psi > \rightarrow U \times < \psi | \times | \psi > \times U^\dagger =$$
$$=< \psi | \times U^\dagger \times (U \times U^\dagger)^{-1} \times U \times | \psi > \times (U \times U^\dagger) =< \hat{\psi} | \hat{\times} | \hat{\psi} > \times \hat{I}, \tag{3.67e}$$

$$H \times | \psi > \rightarrow U \times (H \times | \psi >) = (U \times H \times U^\dagger) \times (U \times U^\dagger)^{-1} \times (U \times | \psi >) =$$
$$= \hat{H} \hat{\times} | \hat{\psi} >, etc. \tag{3.67f}$$

It should be indicated that *not all Lie-Santilli isoalgebras can be constructed via nonunitary transforms of conventional Lie algebras.* As an illustration, the classification of all possible isotopic $\hat{SU}(2)$ algebras exhibits eigenvalues different then the conventional ones, while only conventional eigenvalues are admitted under nonunitary transforms (see Refs. (28,33) for brevity).

3.11: Isorelativity and its isodual. Special relativity is generally presented by contemporary academia as providing a descriptions of all infinitely possible relativistic systems existing in the universe. In Section 2 we have shown that special relativity *cannot* provide a consistent classical description of point antiparticles moving in vacuum. The content of this section establishes that special relativity *cannot* be exactly valid for extended particles and antiparticles moving within physical media. In the next section we shall show that special relativity *cannot* describe irreversible processes for both matter and antimatter. Finally, in Section 5 we shall show that the complexity of biological systems is immensely beyond the rather limited descriptive capacity of special relativity.

Particularly misleading is the widespread statement of the "universal constancy of the speed of light" because contrary to known experimental evidence that *the speed of light is a a local variable depending on the medium in which it propagates,* with well known expression $c = c_o/n$, where c_o is the speed of light in vacuum and n is the familiar index of refraction possessing a rather complex functional dependence on frequencies ω, the density of the medium d, and other variables, $n = n(\omega, d, ...)$.

For the evident intent of salvaging the desired universality of special relativity, speeds $c < c_o$ have been interpreted until recently by reducing the propagation of light within a physical medium to the propagation of photons in vacuum scattering from atom to atom. However, such a reduction is not evidently applicable to the propagation within physical media of radio waves with wavelength of the order of one meter. The same reduction also fails to provide a quantitative interpretation of the dependence of the speed on the frequency, as visible by the naked eye in Newton's spectral decomposition of light. In any case, the reduction of light to photons scattering among atoms has been definitely disproved by the recent experimental evidence of speeds $c > c_o$ occurring within special guides or within media of high density (see Ref. (120) and literature quoted therein).

An illustration of the *inapplicability* (and not of the "violation") of special relativity within physical media is given by the propagation of light and particles in water, where the speed of light is of the order of $c = 2 \times c_o/3$ while electrons can propagate with speeds *bigger* than c, resulting in the emission of the Cerenkov light. If the local speed of light c is assumed as the universal invariant, then the propagation of electrons at speeds $v > c$ is a violation of the principle of causality. If the speed of light *in vacuum* c_o is assumed as the universal invariant *in water,* there is the violation of the relativistic law of addition of two speeds of light c because it *does not* yield the local speed of light c, and there is the violation of other basic axioms of special relativity (see monograph (55) for additional problematic aspects).

It should be also indicated that, *when applied to the propagation of light and particles within physical media, special relativity activates the catastrophic inconsistencies of Theorem 3.1.* This is due to the fact that *the transition from the speed of light in vacuum to that within physical media requires a noncanonical or nonunitary transform.* This point can be best illustrated by using the metric originally proposed by Minkowski, which can be written $\eta = Diag.(1, 1, 1, -c_o^2)$. Then, the transition from c_o to $c = c_o/n$ in the metric can only be achieved via a noncanonical or nonunitary transform

$$\eta = Diag.(1, 1, 1, -c_o^2) \rightarrow \hat{\eta} = Diag.(1, 1, 1, -c_o/n^2) = U \times \eta \times U^\dagger, \tag{3.68a}$$

$$U \times U^\dagger = Diag.(1, 1, 1, 1/n^2) \neq I. \tag{3.68b}$$

An invariant resolution of the above inconsistencies and limitations has been provided by the lifting of special relativity into a new formulation today known as *isorelativity,* or *Lorentz-Poincaré-Einstein-Santilli isorelativity,* where the term "isorelativity" stands to indicate that the principle of relativity applies on isospacetime over isofields, and not on its projection on ordinary spacetime. Also, the additional characterization of "special" is redundant because, as review below, *isorelativity achieves a geometric unification of special and general relativities.* In this section we outline the isotopies of special relativity, while the inclusion of classical and quantum gravity is done in Section 3.13.

Isorelativity was first proposed by R. M. Santilli in Ref. (26) of 1983 via the first invariant formulation of *iso-Minkowskian spaces* and related *iso-Lorentz symmetry.* The studies were then continued in: Ref. (11) of 1985 with the first isotopies of the rotational symmetry; Ref. (28) of 1993 with the first isotopies of the SU(2)-spin symmetry; Ref. (29) of 1993) with the first isotopies of the Poincaré symmetry; and Ref. (33) of 1998 with the first isotopies of the SU(2)-isospin symmetries, Bell's inequalities and local realist. The studies were then completed with memoir (15) of 1998) presenting a comprehensive formulation of the iso-Minkowskian geometry, including its formulation via the mathematics of the Riemannian geometry (such iso-Christoffel's symbols, isocovariant derivatives, etc.).

Numerous independent studies on isorelativity are available in the literature (see, e.g., Refs. (63-68) and [8-11]), such as: Aringazin's proof (192) of the direct universality of the Lorentz-Poincaré-Santilli isosymmetry for all infinitely possible spacetimes with signature $(+,+,+,-)$; Mignani's exact representation (118) of the large difference in cosmological redshifts between quasars and galaxies when physically connected; the exact representation of the anomalous behavior of the meanlifes of unstable particles with speed by Cardone et al (110,11); the exact representation of the experimental data on the Bose-Einstein correlation by Santilli (112) and Cardone and Mignani (113); the invariant and exact validity of the iso-Minkowskian geometry within the hyperdense medium in the interior of hadrons by Arestov et al. (120); the first exact representation of molecular features by Santilli and Shillady (125,126); and numerous others.

Evidently we cannot review isorelativity in the necessary details to avoid a prohibitive length. Nevertheless, to achieve minimal self-sufficiency of this presentation, it is important to outline at least its main structural lines.

The central notion of isorelativity is the lifting of the basic unit of the Minkowski space and of the Poincaré symmetry, $I = Diag.(1,1,1,1)$, into a 4×4-dimensional, nowhere singular and positive-definite matrix $\hat{I} = \hat{I}_{4\times 4}$ with an unrestricted functional dependence on local spacetime coordinates x, speeds v, frequency ω, wavefunction ψ, its derivative $\partial\psi$, etc.,

$$I = Diag.(1,1,1) \rightarrow \hat{I}(x, v, \omega, \psi, \partial\psi, ...) = 1/\hat{T}(x, v, \omega, \psi, \partial\psi, ...) > 0. \tag{3.69}$$

Isorelativity can then be constructed via the method of Section 3.10, namely, by assuming that the basic noncanonical or nonunitary transform coincides with the above isounit (where the diagonalization is permitted by its Hermiticity)

$$U \times U^\dagger = \hat{I} = Diag.(g_{11}, g_{22}, g_{33}, g_{44}), g_{\mu\mu} = g_{\mu\mu}(x, v, \omega, \psi, \partial\psi, ...) > 0, \mu = 1,2,3,4, \tag{3.70}$$

and then subjecting the *totality* of quantities and their operation of special relativity to the above transform.

Let $M(x, \eta, R)$ be the Minkowski space with local coordinates $x = (x^\mu)$, metric $\eta = Diag.(1,1,1,-1)$ and invariant $x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu_\times I \in R$. The fundamental space of isorelativity is the *Minkowski isospace* (26,15) and related topology (14), $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ characterized by the dual lifting of the basic unit (and related field) and the *inverse* lifting of the metric as per rules (3.35)

$$I = Diag.(1,1,1,1) \rightarrow U \times I \times U^\dagger = \hat{I} = 1/\hat{T}, \tag{3.71a}$$

$$\eta = Diag.(1,1,1,-1) \times I \rightarrow (U^{\dagger -1} \times \eta \times U^{-1}) \times \hat{I} = \hat{\eta} =$$
$$= \hat{T} \times \eta = Diag.(g_{11}, g_{22}, g_{33}, -g_{44}) \times \hat{I}, \tag{3.71b}$$

with consequential isotopy of the basic invariant

$$x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu_\times I \in R \rightarrow$$

$$\to U \times x^2 \times U^\dagger = \hat{x}^{\hat{2}} = (\hat{x}^\mu \hat{\times} \hat{m}_{\mu\nu} \times x^\nu_\times I \in R, \tag{3.72}$$

whose projection in conventional spacetime can be written

$$\hat{x}^{\hat{2}} = [x^\mu \times \hat{\eta}_{\mu\nu}(x, v, \omega, \psi, \partial\psi, ...) \times x^\nu_] \times \hat{I}, \tag{3.73}$$

The nontriviality of the above lifting is illustrated by the fact that *Minkowski-Santilli isospaces include as particular spaces all possible spacetimes, such as the Riemannian, Finslerian, non-Desarguesian and any other space with the signature* $(+, +, +, -)$. Moreover, the iso-Minkowskian metric $\hat{\eta}$ depends explicitly on the local coordinates. Therefore, *the Minkowski-Santilli isogeometry requires for its formulation the isotopy of all tools of the Riemannian geometry,* such as the iso-Christoffel symbols, isocovariant derivative, etc. (see for brevity Ref. (15)). Despite that, one should keep in mind that, in view of the positive-definiteness property (34.79), *the Minkowski-Santilli isogeometry coincides at the abstract level with the conventional Minkowski geometry, thus having a null isocurvature* (because of the basic mechanism of deforming the metric η by the amount $\hat{T}(x, ...)$ while deforming the basic unit of the inverse amount $\hat{I} = 1/\hat{T}$).

It should be also noted that, following the publication in 1983 of Ref. (26), numerous papers on "deformed Minkowski spaces" have appeared in the physical and mathematical literature (generally without a quotation of their origination in Ref. (29)). These "deformations" are formulated via conventional mathematics and, consequently, they all suffer of the catastrophic inconsistencies of Theorem 3.1. By comparison, isospaces are formulated via isomathematics and, therefore, they resolve the inconsistencies of Theorem 3.1, as shown in Section 3.9. This illustrates the necessity of lifting the basic unit and related field jointly with any noncanonical lifting of canonical metrics.

Let $P(3.1)$ be the conventional Poincaré symmetry with the well known ten generators $J_{\mu\nu}, P_\mu$ and related commutation rules. The second basic tool of isorelativity is the *Poincaré-Santilli isosymmetry* $\hat{P}(3.1)$ which can be constructed via the isotheory of Section 3.5, resulting in the isocommutation rules (26,29)

$$[J_{\mu\nu} \hat{,} J_{\alpha\beta}] = i \times (\hat{\eta}_{\nu\alpha} \times J_{\beta\mu} - \hat{\eta}_{\nu\alpha} \times J_{\beta\nu} - \hat{\eta}_{nu\beta} \times J_{\alpha\mu} + \hat{\eta}_{\mu\beta} \times J_{\alpha\nu}), \tag{3.74a}$$

$$[J_{\mu\nu} \hat{,} P_\alpha] = i \times (\hat{\eta}_{\mu\alpha} \times P_\nu - \hat{\eta}_{\nu\alpha} \times P_\mu), [P_\mu \hat{,} P\nu] = 0, \tag{3.74b}$$

. where we have followed the general rule of the Lie-Santilli isotheory according to which isotopies leave observables unchanged (since Hermiticity coincides with iso-Hermiticity) and merely change the *operations* among them.

Isorelativistic kinematics is then based on the following two iso-invariants:

$$P^{\hat{2}} = P_\mu \hat{\times} P^\mu = P^\mu \times \eta_{\mu\nu} \times P^\nu = P_k \times g_{kk} \times P_k - p_4 \times g_{44} \times P_4, \tag{3.75a}$$

$$W^{\hat{2}} = W_\mu \hat{\times} W^\mu, W_\mu = \hat{\epsilon}_{\mu\alpha\beta\rho} \hat{\times} J^{\alpha\beta} \hat{\times} P^\rho. \tag{3.75b}$$

Since $\hat{I} > 0$, it is easy to prove Lemma 3.3, namely, that *the Poincaré-Santilli isosymmetry is isomorphic to the conventional symmetry.* It then follows that *the isotopies* *increase dramatically the arena of applicability of the Poincaré symmetry, from the sole Minkowskian spacetime to all infinitely possible spacetimes.*

To understand the physical, chemical and biological applications outline in this paper, the reader should be aware that *all "particles" considered hereon are assumed to be "isoparticles", that is, irreducible isorepresentation of the Poincaré-Santilli isosymmetry,* namely, particles are assumed to be extended, generally nonspherical and deformable under Hamiltonian and non-Hamiltonian interactions.

Since any interaction imply a renormalization of physical characteristics, it is evident that *the transition form particles to isoparticles, that is. from motion in vacuum to motion within physical media, implies an alteration (called isorenormalization) of all the intrinsic characteristics, such as rest energy, magnetic moment, charge, etc.* As we shall see in Section 3.14, such isorenormalization have permitted the first exact numerical representation of nuclear magnetic moments which had resulted to be impossible for quantum mechanics despite about 75 years of attempts.

The explicit form of the Poincaré-Santilli isotransforms leaving invariant line element (3.73) are given by:

(1) The *isorotations* $\hat{O}(3) : \hat{\mathbf{x}}' = \hat{\Re}(\hat{\theta}) \hat{\times} \hat{\mathbf{x}}, \hat{\theta} = \theta \times \hat{I}_\theta \in \hat{R}_\theta$ (11) which, for isorotations in the (1, 2) isoplane, are given by

$$x^{1'} = x^1 \times cos[\theta \times (g_{11} \times g_{22})^{1/2}] - x^2 \times g_{22} \times g_{11}^{-1} \times sin[\theta \times (g_{11} \times g_{22})^{1/2}], \quad (3.76a)$$

$$x^{2'} = x^1 \times g_{11} \times g_{22}^{-1} \times sin[\theta \times (g_{11} \times g_{22})^{1/2}] + x^2 \times cos[\theta \times (g_{11} \times g_{22})^{1/2}]. \quad (3.76b)$$

For the general expression in three dimensions interested reader can inspect Ref. (55) for brevity.

Note that, since $\hat{O}(3)$ is isomorphic to $O(3)$, Ref. (11) proved that, contrary to a popular belief throughout the 20-th century, *the rotational symmetry remains exact for all possible signature-preserving (+,+,+) deformations of the sphere,* of course, when treated with the appropriate mathematics.

The above reconstruction of the exact rotational symmetry can be geometrically visualized by the fact that *all possible signature-preserving deformations of the sphere are perfect spheres in isospace called isosphere.* This is due to the fact that ellipsoidical deformations of the semiaxes $1_k \rightarrow 1/n_k^2$ are compensated on isospaces over isofields by the *inverse* deformation of the related unit $1_k \rightarrow n_k^2$. Therefore, by recalling structure (3.35) of the isoinvariant, on iso-Euclidean space we have the perfect isosphere $\hat{r}^{\hat{2}} = \hat{r}_1^2 + \hat{r}_2^2 + \hat{r}_3^2$ with exact $\hat{O}(3)$ symmetry while its projection on the conventional Euclidean space is the ellipsoid $r_1^2/n_1^2 + r_2^2/n_2^2 + r_3^2/n_3^2$ with broken $O(3)$ symmetry.

(2) The *Lorentz-Santilli isotransforms* $\hat{O}(3.1) : \hat{x}' = \hat{\Lambda}(\hat{v}, ...) \hat{\times} \hat{x}, \hat{v} = v \times \hat{I}_v \in \hat{R}_v$ (26,29) which, for the case of isorotation in the $(3, 4)$ isoplane, can be written

$$x^{1'} = x^1, \quad (3.77a)$$

$$x^{2'} = x^2, \quad (3.77b)$$

$$x^{3'} = x^3 \times cosh[v \times (g_{33} \times g_{44})^{1/2}] -$$

$$-x^4 \times g_{44} \times (g_{33} \times g_{44})^{-1/2} \times sinh[v \times (g_{33} \times g_{44})^{1/2}] =$$
$$= \hat{\gamma} \times (x^3 - \beta \times x^4), \quad (3.77c)$$
$$x^{4\prime} = -x^3 \times g_{33} \times (g_{33} \times g_{44})^{-1/2} \times sinh[v(g_{33} \times g_{44})^{1/2}] +$$
$$+ x^4 \times cosh[v \times (g_{33} \times g_{44})^{1/2}] =$$
$$= \hat{\gamma} \times (x^4 - \hat{\beta} \times x^3), \quad (3.77d)$$
$$\hat{\beta} = \frac{v_k \times g_{kk} \times v_k}{c_o \times g_{44} \times c_o}, \hat{\gamma} = \frac{1}{(1 - \hat{\beta}^2)^{1/2}}. \quad (3.77e)$$

For the general expression interested readers can inspect Ref. (55).

Ref. (26) proved that, contrary to another popular belief throughout the 20-th century, *the Lorentz symmetry remains exact for all possible signature preserving (+,+,+,1) deformations of the Minkowski space,* of course, when treated with the appropriate mathematics.

The above exact reconstruction of the Lorentz symmetry can be geometrically visualized by noting that the light cone $x_3^2 - c_o^2 \times t^2 = 0$ can only be formulated in vacuum while within physical media we have the generic hyperboloid $r_3^2 - c_o^2 \times t^2/n^2(\omega, ...) = 0$. However, it is an instructive exercise for interested readers to prove that *the isolight cone (that is, the light cone on isospace over isofields) is the perfect cone* $\hat{r}_3^2 - c_o^2 \times \hat{t} = 0$ with the exact symmetry $\hat{O}(3.1)$ while its projection on conventional space is given by $r_3^2 - c_o^2 \times t^2/n^2(\omega, ...) = 0$ with broken Lorentz symmetry.

(3) The *isotranslations* $\hat{\mathcal{T}}(4) : \hat{x}' = \hat{\mathcal{T}}(\hat{a}, ...)\hat{]} \times x = \hat{x} + \hat{A}(\hat{a}, x, ...), \hat{a} = a \times \hat{I}_a \in \hat{R}_a$ which can be written

$$x^{\mu\prime} = x^\mu + A^\mu(a, ...), \quad (3.78a)$$
$$A^\mu = a^\mu(g_{\mu\mu} + a^\alpha \times [g_{,i\mu} \hat{,} P_\alpha]/1! + ...), \quad (3.78b)$$

where there is no summation on the μ indices.

Note that the isotranslations are highly nonlinear (thus non-inertial) in conventional spacetime although they are isolinear (thus inertial) in isospace. This illustrates the reason why conventional notion of relativity are solely applicable in spacetime, thus illustrating the reason of the name "isorelativity."

(4) The novel *isotopic invariance* $\hat{\mathcal{I}} : \hat{x}' = \hat{w} \hat{\times} \hat{x} = w \times \hat{x}, \hat{I}' = w \times \hat{I}$, where w is a constant (29),

$$\hat{I} \rightarrow \hat{I}' = \hat{w} \hat{\times} \hat{I} = w \times \hat{I} = 1/\hat{T}', \quad (3.79a)$$
$$\hat{x}^{\hat{2}} = (x^\mu \times \hat{\eta}_{\mu\nu} \times x^\nu_) \times \hat{I} \equiv \hat{x}'^{\hat{2}} = [x^\mu \times (w^{-1} \times \hat{\eta}_{\mu\nu}) \times x^\nu_] \times (w \times \hat{I}), \quad (3.79b)$$

Therefore, the Poincaré-Santilli isosymmetry can be written

$$\hat{P}(3.1) = \hat{O}(3.1) \hat{\times} \hat{\mathcal{T}}(4) \hat{\times} \hat{\mathcal{I}} \quad (3.80)$$

thus having *eleven (rather than ten) dimensions* with parameters $\theta^k, v^k, a^\mu, w, k = 1, 2, 3, \mu = 1, 2, 3, 4$, the 11-th dimension being characterized by invariant (3.78). Note that, contrary to popular beliefs, *the conventional Poincaré symmetry is also eleven dimensional* since invariance (3.78) also holds for conventional spacetime.

The simplest possible realization of the above formalism for isorelativistic kinematics can be outlined as follows (see Section 3.13 for the isogravitational realization). The first application of isorelativity is that of providing *an invariant description of locally varying speeds of light propagating within physical media.* For this purpose a realization of isorelativity requires the knowledge of the *density* of the medium in which motion occurs.

The simplest possible realization of the fourth component of the isometric is then given by the function $g_{44} = n_4^2(x, \omega, ...)$ normalized to the value $n_4 = 1$ for the vacuum (note that the density of the medium in which motion occur *cannot* be described by special relativity). Representation (3.68) then follows with invariance under $\hat{P}(3.1)$.

In this case the quantities $n_k, k = 1, 2, 3$, represent the inhomogeneity and anisotropy of the medium considered. For instance, if the medium is homogeneous and isotropic (such as water), all metric elements coincide, in which case

$$\hat{I} = Diag.(g_{11}, g_{22}, g_{33}, g_{44}) = n_4^2 \times Diag.(1, 1, 1, 1), \tag{3.81a}$$

$$\hat{x}^{\hat{2}} = \frac{x^2}{n_4^2} \times n_4^2 \times I \equiv x^2, \qquad (3.81b).$$

thus confirming that *isotopies are hidden in Minkowskian axioms,* and this may be a reason why they have nog been discovered until recently.

Next, isorelativity has been constricted for the invariant description of systems of extended, nonspherical and deformable particles under Hamiltonian and non-Hamiltonian interactions. Practical applications then require the knowledge of the actual shape of the particles considered, here assumed for simplicity as being spheroidal ellipsoids with semiaxes n_1^2, n_2^2, n_3^2. Note that them minimum number of constituents of a closed non-Hamiltonian system is two. In this case we have shapes represented with $n_{\alpha k}, \alpha = 1, 2, , ..., n$. Applications finally require the identification of the nonlocal interactions, e.g., whether occurring on an extended *surface* or *volume.* As an illustration, two spinning particles denoted 1 and 2 in condition of deep mutual penetration and overlapping of their wavepackets (as it is the case for valence bonds), can be described by the following Hamiltonian and total; isounit

total isounit

$$H = \frac{p_1 \times p_1}{2 \times m_1} + \frac{p_2 \times p_2}{2 \times m_2} + V(r), \tag{3.82a}$$

$$\hat{I}_{Tot} = Diag.(n_{11}^2, n_{12}^2, n_{13}^2, n_{14}^2) \times Diag.(n_{21}^2, n_{22}^2, n_{23}^2, n_{24}^2) \times$$
$$\times e^{N \times (\hat{\psi}_1/\psi_1 + \hat{\psi}_2/\psi_2) \times \int \hat{\psi}_{1\uparrow}(r)^\dagger \times \hat{\psi}_{2\downarrow}(r) \times dr^3}, \tag{3.82b}$$

where N is a constant. Note the nonlinearity in the wavefunctions, the nonlocal-integral character and the lack of representation of all the above features via a Hamiltonian. From the above examples interested readers can then represent any other closed non-Hamiltonian systems.

The third important part of isorelativity is given by the following isotopies of conventional relativistic axioms which, for the case of motion along the third axis, can be written (29):

ISOAXIOM I. The projection in our spacetime of the maximal causal invariant speed is given by:

$$V_{Max} = c_o \times \frac{g_{44}^{1/2}}{g_{33}^{1/2}} = c_o \frac{n_3}{n_4} = \frac{c}{n_3}. \tag{3.83}$$

This isoaxioms resolves the inconsistencies of special relativity recalled earlier for particles and electromagnetic waves propagating in water. In fact, water is homogeneous and isotropic, thus requiring that $g_{44} = g_{33} = 1/n^2$, where n is the index of refraction. In this case the maximal causal speed for a massive particle is c_o as experimentally established, e.g., for electrons, while the local speed of electromagnetic waves is $c = c_o/n.$, as also experimentally established.

Note that such a resolution requires the abandonment of the speed of *light* as the maximal causal speed for motion within physical media, and its replacement with the maximal causal speed of *particles*. It happens that in vacuum these two maximal causal speeds coincide. However, even in vacuum the correct maximal causal speed remains that of particles and *not* that of light, as generally believed. At any rate, physical media are generally opaque to light but generally not to particles. Therefore, the assumption as the maximal causal speed as that of light which cannot propagate within the medium considered would be evidently vacuous.

It is an instructive exercise for the interested readers to prove that *the maximal causal speed of particles on isominkowski space over an isofield remains c_o.*

ISOAXIOM II. The projection in our spacetime of the isorelativistic addition of speeds within physical media is given by:

$$v_{Tot} = \frac{v_1 + v_2}{1 + \frac{v_1 \times g_{33} \times v_2}{c_o \times g_{44} \times c_o}} = \frac{v_1 + v_2}{1 + \frac{v_1 \times n_4^2 \times v_2}{c_o \times n_3^2 \times c_o}} \tag{3.84}$$

We have again the correct occurrence that *the sum of two maximal causal speeds in water, $V_{max} = c_o \times (n_3/n_4)$, yields the maximal causal speed in water,* as the reader is encouraged to verify. Note that the such a result is impossible for special relativity. Note also that *the isorelativistic sum of two speeds of lights in water, $c = c_o/n$, does not yield the speed of light in water,* thus confirming that the speed of light within physical media, assuming that they are transparent to light, is not the fundamental maximal causal speed.

ISOAXIOM III. The projection in our spacetime of the isorelativistic laws of dilation of time t_o and contraction of length ℓ_o and the variation of mass m_o with speed are given by:

$$t = \hat{\gamma} \times t_o, \ell = \hat{\gamma}^{-1} \times \ell_o, m = \hat{\gamma} \times m_o. \tag{3.85}$$

Note that in water these values coincide with the relativistic one as it should be since particles such as the electrons have in water the maximal causal speed c_0. Note again the necessity of avoiding the interpretation of the local speed of light as the maximal local causal speed. Note that the mass diverges at the maximal local causal speed, but *not* at the local speed of light..

ISOAXIOM IV. The projection in our spacetime of the iso-Doppler law is given by (for 90^o *angle of aberration):*

$$\omega = \hat{\gamma} \times \omega_o. \tag{3.86}$$

This isorelativistic axioms permits an exact, numerical and invariant representation of the large differences in cosmological redshifts between quasars and galaxies when physically connected. In this case light simply exit the huge quasar chromospheres already redshifted due to the decrease of the speed of light, rather than the speed of the quasars (118).

Isoaxiom IV also permits a numerical interpretation of the internal blue- and red-shift of quasars due to the dependence of the local speed of light on its frequency. Finally, Isoaxiom IV predicts that *a component* of the predominance toward the red of sunlight at sunset is of iso-Doppler nature in view of the bigger decrease of the speed of light at sunset as compared to the same speed at the zenith (evidently because of the travel within a comparatively denser atmosphere).

ISOAXIOM V. The projection in our spacetime of the isorelativistic law of equivalence of mass and energy is given by:

$$E = m \times c_o^2 \times g_{44} = m \times \frac{c_o^2}{n_4^2}. \tag{3.87}$$

Among various applications, *Isoaxiom V removes any need for the "missing mass" in the universe.* This is due to the fact that all isotopic fits of experimental data agree on values $g_{44} \gg 1$ within the hyperdense media in the interior of hadrons, nuclei and stars (55,120). As a result, Isoaxiom V yields a value of the total energy of the universe dramatically bigger than that believed until now under the assumption of the universal validity of the speed of light in vacuum. For other intriguing applications, e.g., for the rest energy of hadronic constituents, we refer the interested reader to monographs (55,61).

The *isodual isorelativity* for the characterization of antimatter can be easily constructed via the isodual map of Section 2, and its explicit study is left to the interested reader for brevity.

3.12: Isorelativistic Hadronic Mechanics and its isodual. The isorelativistic extension of nonrelativistic hadronic mechanics is readily permitted by the Poincaré-Santilli isosymmetry. In fact, iso-invariant (3.75a) implies the following *iso-Gordon equation* on $\hat{\mathcal{H}}$ over $\hat{C}$

$$\hat{p}_\mu \hat{\times} |\hat{\psi}> = -\hat{i} \hat{\times} \hat{\partial}_\mu |\hat{\psi}> = -i \times \hat{I}^\nu_\mu \times \partial_\nu |\hat{\psi}>, \tag{3.88a}$$

$$(\hat{p}_\mu \hat{\times} \hat{p}^\mu + \hat{m}_o^2 \hat{\times} \hat{c}^4) \hat{\times} |\hat{\psi}> = (\hat{\eta}^{\alpha\beta} \times \partial_\alpha \times \partial_\beta + m_o^2 \times c^4) \times |\hat{\psi}> = 0. \tag{3.88b}$$

The linearization of the above second-order isoinvariant into the *iso-Dirac equation* has been studied in detail in Ref. (29) as well as by several other authors (although generally without the use of isomathematics, thus losing the invariance). By recalling the correct structure (2.34) of Dirac's equation as the Kronecker product of a spin 1/2 massive particle and its antiparticle, the iso-Dirac's equation is formulated on the total isoselfadjoint isospace and related isosymmetry

$$\hat{M}^{Tot} = [\hat{M}^{orb}(\hat{x}, \hat{\eta}, \hat{R}) \times \hat{S}^{spin}(2)] \times [\hat{M}^{dorb}(\hat{x}^d, \hat{\eta}^d, \hat{R}^d) \times \hat{S}^{dspin}(2)] = \hat{M}^{dTot}, \qquad (3.89a)$$

$$\hat{S}^{Tot} = \hat{P}(3.1) \times \hat{P}^d(3.1) = \hat{S}^{dTot}, \qquad (3.89b)$$

and can be written (29)

$$[\hat{\gamma}^\mu \hat{\times} (\hat{p}_\mu - \hat{e} \hat{\times} \hat{A}_\mu) + \hat{i} \hat{\times} \hat{m}] \hat{\times} |\phi(x) >= 0, \qquad (3.90a)$$

$$\hat{\gamma}^\mu = g^{\mu\mu} \times \gamma^\mu \times \hat{I} \qquad (3.90b)$$

where the γ's are the conventional Dirac matrices. Note the appearance of the isometric elements directly in the structure of the gamma matrices and their presence also when the equation is projected in the conventional spacetime.

A realization via the iso-Dirac equation of the Poincaré-Santilli isosymmetry with isocommutators (3.74) is given by (29)

$$J_{\mu\nu} = (S_k, L_{k4}), P_\mu, \qquad (3.91a)$$

$$S_k = (\hat{\epsilon}_{kij} \hat{\times} \hat{\gamma}_i \hat{\times} \hat{\gamma}_j)/2, L_{k4} = \hat{\gamma}_k \hat{\times} \hat{\gamma}_4 /2, P_\mu = \hat{p}_\mu \qquad (3.91b)$$

The notion of "isoparticle" can be best illustrated with the above realization because it implies that,*in the transition from motion in vacuum (as particles have been solely detected and studied until now) to motion within physical media, particles generally experience the alteration, called "mutation," of all intrinsic characteristics,* as illustrated by the following isoeigenvalues which are implied by isocommutation rules (3.74),

$$\hat{S}^{\hat{2}} \hat{\times} |\hat{\psi} >= \frac{g_{11} \times g_{22} + g_{22} \times g_{33} + g_{33} \times g_{11}}{4} \times |\hat{\psi} >, \qquad (3.92a)$$

$$\hat{S}_3 \hat{\times} |\hat{\psi} >= \frac{(g_{11} \times g_{22})^{1/2}}{2} \times |\hat{\psi} > . \qquad (3.92b)$$

The mutation of spin then implies a necessary mutation of the intrinsic magnetic moment which is given by (29)

$$\tilde{\mu} = (\frac{g_{33}}{g_{44}})^{1/2} \times \mu, \qquad (3.93)$$

where μ is the conventional magnetic moment for the same particle when in vacuum. The mutation of the rest energy and of the remaining characteristics has been identified before via the isoaxioms.

The construction of the isodual isorelativistic hadronic mechanics is left to the interested reader by keeping in mind that the iso-Dirac equation is isoselfdual as the conventional equation.

To properly understand the above results, one should keep in mind that *the mutation of the intrinsic characteristics of particles is solely referred to the constituents of a hadronic bound state under condition of mutual penetration of their wavepackets (such as one hadronic constituent) under the condition of recovering conventional characteristics for the hadronic bound state as a whole (the hadron considered),* much along the original Newtonian subsidiary constrains on non-Hamiltonian forces, Eqs. (3.42b)-(3.42d).

The reader should also keep in mind that, at this kinematical level prior to the introduction of gravity, $g_{44} = 1/n_4^2$ represent *the density of the medium in which motion occurs,* normalized to the value $g_{44} = 1, n_4 = 1$ for the vacuum. Also, the *inhomogeneity of the medium* is represented by the functional dependence of its density, e.g., from the radial distance r and other variables, $g_{44} = g_{44}(r, ...)$. The *anisotropy of the medium* is represented by g_{33}, e.g., for the case of the spheroidal ellipsoid for which $g_{11} = g_{22} \neq g_{33}$.

Finally, isotopic invariance (3.79) implies the capability of rescaling the radius of a sphere. Therefore, for the case of the perfect sphere we can always have $g_{11} = g_{22} = g_{33} = g_{44}$ in which case the magnetic moment is not mutated. These results recover conventional classical knowledge according to which *the alteration of the shape of a charged and spinning body implies the necessary alteration of its magnetic moment.*

It should be also stressed that *the above mutations violate the unitary condition when formulated on a conventional Hilbert spaces, with consequential catastrophic inconsistencies.* As an illustration, the violation of causality and probability law has been established for all eigenvalues of the angular momentum M different than the quantum spectrum $M^2 \times |\psi >= \ell(\ell + 1) \times |\psi >, \ell = 0, 1, 2, 3, ...$. As a matter of fact, these inconsistencies are the very reason why the mutations of internal characteristics of particles for bound states at short distances could not be admitted within the framework of quantum mechanics.

By comparison, hadronic mechanics has been constructed precisely to recover unitarity on iso-Hilbert spaces over isofields, thus permitting an invariant description of internal mutations of the characteristics of the constituents of hadronic bound states, while recovering conventional features for states as a whose.

As we shall indicate at the end of this section, far from being mathematical curiosities, the above mutations imply basically new structure models of hadrons, nuclei and stars, with consequential, new clean energies and fuels. These new advances were prohibited by quantum mechanics precisely because of the preservation of the *intrinsic* characteristics of the constituents in the transition from bound states at large mutual distance, for which no mutation is possible, to the bound state of the same constituents in condition of mutual penetration, in which case mutations have to be admitted in order to avoid the replacement of a scientific process with unsubstantiated personal beliefs one way or the other.

The best illustration of the iso-Dirac equation is, therefore, that for which it was constructed (30), to describe the transition for the electron in the hydrogen atom, to the same electron when compressed in the hyperdense medium in the interior of the proton, namely, to achieve a quantitative and invariant representation of the synthesis of the neutron according to Rutherford as a "hydrogen atom compressed in the core of a star."

If special relativity, relativistic quantum mechanics and the conventional Dirac equation are assumed to be exactly valid also for the motion of the electron within the hyperdense medium in the interior of the proton, the neutron *cannot* be a bound state of a proton and an electron at short distances, thus mandating the assumption of undetectable constituents which cannot be produced free, as well known.

One of the most important results of hadronic mechanics has been the proof at the nonrelativistic (214) and relativistic level (30) that a hadronic bound state of an *isoprotons* and an *isoelectron* represents *all* characteristics of the neutron, including its rest energy, spin, charge, parity, charge radius, anomalous magnetic moment and spontaneous decay.

The societal implications of the above alternative are such to require the surpassing of traditional academic interests on pre-established doctrines. In fact, no new energy is conceivably possible under the assumption of the exact validity within a hadron of the Minkowski geometry, the special relativity and relativistic quantum mechanics, with consequential hadronic constituents which cannot be produced free. On the contrary, the assumption that isorelativity within the hyperdense medium inside hadrons implies that the hadronic constituents are indeed produced free in the spontaneous decay and, therefore, they can indeed be stimulated to decay, thus implying basically new energies (58).

3.13: Isogravitation, iso-grand-unification and isocosmology. There is no doubt that the classical and operator formulations of gravitation on a curved space has been the most controversial theory of the 20-th century because of an ever increasing plethora of problematic aspects which have remained basically unresolved due to the lack of their acknowledgment, let alone their resolution, by leading research centers in the field (see, for instance, H. E. Wilhelm (220) and references quoted therein).

One of the reason why special relativity in vacuum has a majestic axiomatic consistence is its *invariance* under the poincaré symmetry. Recent studies have shown that the formulation of gravitation on a curved space or, equivalently, rathe formulation of gravitation based on as "covariance," is necessarily noncanonical at the classical level and nonunitary at the operator level, thus suffering of all catastrophic inconsistencies of Theorem 3.1 (45,46). These catastrophic inconsistencies can only be resolved via a new conception of gravity based on a *universal invariance,* rather than covariance.

Additional studies have identified profound axiomatic incompatibilities between gravitation on a curved space and electroweak interactions. These incompatibilities have resulted to be responsible for the lack of achievement of an axiomatically consistent grand unification since Einstein's times (32,35,37), among which we mention:

1) Electroweak theories are based on *invariance* while gravitation is not;

2) Electroweak theories are flat in their axioms while gravitation is not; and

3) Electroweak theories are *bona fide* field theories, thus admitting positive and negative energy solutions, while gravitation can only admit positive energies.

No knowledge of isotopies can be claimed without a knowledge that isorelativity has been constructed also to resolve at least some of the controversies on gravitation. The fundamental requirement is *the abandonment of the formulation of gravity on a Rieman-*

nian space and its formulation instead on an iso-Minkowskian space (15) via the following basic steps:

I) Factorization of any given Riemannian metric $g(x)$ into a nowhere singular and positive-definite 4×4 matrix $\hat{T}(x)$ times the Minkowski metric η,

$$g(x) = \hat{T}_{grav}(x) \times \eta; \tag{3.94}$$

II) Assumption of the inverse of $\hat{T}_{grav}$ as the fundamental unit of the theory,

$$\hat{I}_{grav}(x) = 1/\hat{T}_{grav}(x); \tag{3.95}$$

III) Submission of the totality of the Minkowski space and relative symmetries to the noncanonical/nonunitary transform

$$U(x) \times I^{\dagger}(x) = \hat{I}_{grav}. \tag{3.96}$$

The above procedure yields the isominkowskian spaces and related geometry $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$, (15), resulting in a new conception of gravitation, called *isogravity,* with the following main features (15,32,35,37,55):

i) Isogravity is characterized by a universal *symmetry* (and not a covariance), the Poincaré-Santilli isosymmetry $\hat{P}(3.1)$ for the gravity of matter with isounit $\hat{I}_{grav}(x)$ with the isodual isosymmetry $\hat{P}^d(3.1)$ for the gravity of antimatter, and the isodual symmetry $\hat{P}(3.1) \times \hat{P}^d(3.1)$ for the gravity of matter-antimatter systems;

ii) All conventional field equations, such as the Einstein-Hilbert and other field equations, can be identically formulated via the Minkowski-Santilli isogeometry since the latter preserves all the tools of the conventional Riemannian geometry, such as the Christoffel's symbols, covariant derivative, etc. (15);

iii) Isogravitation is isocanonical; at the classical level and isounitarity at the operator level, thus resolving the catastrophic inconsistencies of Theorem 3.1;

iv) An axiomatically consistent operator version of gravity always existed and merely creeped in un-noticed through the 20-th century because gravity is embedded where nobody looked for, in the *unit* of relativistic quantum mechanics, and it is given by isorelativistic hadronic mechanics as in Eqs. (3.88) and (3.90).

v) The basic feature permitting the above advances is the abandonment of curvature for the characterization of gravity (namely, curvAture characterized by metric $g(x)$ referred to the unit I) and its replacement with *isoflatness,* namely, the verification of the axioms of flatness in isospace, while preserving conventional curvature in its projection on conventional spacetime (or, equivalently, curvature characterized by the $g(x) = \hat{T}_{\text{grav}}(x) \times \eta$ referred to the isounit $\hat{I}_{\text{grav}}(x)$ in which case curvature becomes null due to the interrelation $\hat{I}_{\text{grav}}(x) = 1/\hat{T}_{\text{grav}}(x)$) (15).

A resolution of numerous controversies on classical formulations of gravity then follow from the above main features, such as: the resolution of the century old controversy on the lack of existence of consistent total conservation laws for gravitation on a Riemannian space (which controversy is resolved under the universal $\hat{P}(3.1)$ symmetry by mere visual verification that the generators of the conventional and isotopic Poincaré symmetry are the same, since they represent conserved quantities in the absence and in the preserve of gravity); the controversy on the fact that gravity on a Riemannian space admits a well defined "Euclidean," but not ""Minkowskian" limit (which controversy is trivially resolved by isogravity via the limit $\hat{I}_{\text{grav}}(x) \to I$); and others.

A resolution of the controversies on quantum gravity can be seen from the property that relativistic hadronic mechanics *is* a quantum formulation of gravity whenever $\hat{T} = \hat{T}_{\text{grav}}$ which is as axiomatically consistent as the conventional relativistic quantum mechanics because the two formulations coincide, by construct, at the abstract, realization-free level. As an illustration, whenever $\hat{T}_{\text{grav}} = Diag.(g_{11}, g_{22}, g_{33}, g_{44})$, the iso-Dirac equation (3.90) provides a direct rtepresentation of the conventional electrom-nagnetic interactions experienced by an electron, represented by the fevot potential A_μ, plus gravitational interactions represented by the isogamma matrices.

Once curvature is abandoned in favor of the broader isoflatness, the axiomatic incompatibilities existing between gravity and electroweak interactions are resolved because: isogravity possesses, at the abstract level, the *same* Poincaré invariance of electroweak interactions; isogravity can be formulated on the *same* flat isospace of electroweak theories; and isogravity admits positive and negative energies in the *same* way as it occurs for electroweak theories. An axiomatically consistent *iso-grand-unification* then follows (32,35).

Note that the above grand-unification requires the prior *geometric unification of the special and general relativities,* which is achieved precisely by isorelativity and its underlying iso-Minkowskian geometry. In fact, special and general relativities are merely differentiated in isospecial relativity by the explicit realization of the unit. In particular, *black holes are now characterized by the zeros of the isounit* (55)

$$\hat{I}_{\text{grav}}(x) = 0. \tag{3.97}$$

The above formulation recovers all conventional results on gravitational singularities, such as the singularities of the Schwarzschild's metric, since they are all described by the gravitational content $\hat{T}_{\text{grav}}(x)$ of $g(x) = \hat{T}_{\text{grav}}(x) \times \eta$, since η is flat.

This illustrates again that *all conventional results of gravitation, including experimental verifications, can be reformulated in invariant form via isorelativity.*

Moreover, the problematic aspects of general relativity mentioned earlier refer to the *exterior gravitational problem.* Perhaps greater problematic aspects exist in gravitation on a Riemannian space for *interior gravitational problems,* e.g., because of the lack of characterization of basic features, such as the density of the interior problem, the locally varying character of gravitation, etc. These additional problematic aspects are also resolved by isospecial relativity due to the unrestricted character of the functional dependence of the isometric which therefore permits a direct geometrization of the density, local; variation of the speed of light, etc.

The cosmological implications are also intriguing. In fact, isorelativity permits a new conception of cosmology based on the *universal invariance* $\hat{P}(3.1) \times \hat{P}^d(3.1)$ in which there is no need for the "missing mass" (as indicated earlier), time and the speed of light become local variables, and the detected universe has a dimension considerably *smaller* than that currently believed (because some of the cosmological redshift is due to the decrease of the speed of light in chromospheres, rather than speed of quasars). Also, at the limit case of equal distribution of matter and antimatter in the universe, *isocosmology predicts that the universe has identically null total energy, identically null total time, and identically null other physical characteristics, thus permitting mathematical studies of its creation* (because of the lack of singularities at its formation).

3.14: Experimental verifications and scientific applications. Nowadays, isotopies in general, and the isotopic branch of hadronic mechanics in particular, have clear experimental verifications in classical physics, particle physics, nuclear physics, chemistry, superconductivity, biology and cosmology, among which we quote the following representative verifications:

⋆ The first and only known optimization of the shape of extended objects moving within resistive media via the optimal control theory (14) following the first achievement of the universality of the isoaction (3.50) for all possible resistive forces; ⋆ The axiomatically correct formulation of special relativity in terms of the proper time by T. Gill and his associates (202)-(206);

⋆ The first identification of the connection between Lie-admissibility and supersymmetries by Adler (211);

⋆ The proof by Aringazin (192,197) of the "universality" of Isoaxiom III, namely, its capability of admitting as particualr cases all available anomalous time dilations via different expensions in terms of different quantities and with different truncatiuons;

⋆ The exact representation of the anomalous behavior of the meanlives of unstable particles with speed by Cardone et al (110,11) that to Isoaxioms III of isorelativity;

⋆ The exact representation of the experimental data on the Bose-Einstein correlation by Santilli (112) and Cardone and Mignani (113) under the exact iso-Poincaré symmetry;

⋆ The invariant and exact validity of the iso-Minkowskian geometry within the hyperdense medium in the interior of hadrons by Arestov et al. (120);

⋆ The achievement of an exact confinement of quarks by Kalnay (216) and Kalnay and Santilli (2.17) thanks to incoherence between the external and internal Hilbert spaces;

⋆ The proof by Jannussis and Mignani (186) of the convergence of isotopic perturbative series when conventionally divergent based on the property for all isotopic elements used in actual models $\hat{T} \ll 1$, thus implying that perturbative expansions which are divergent when formulated with the conventional associative product $A \times B$ become convergent when re-expressed in terms of the isoassociative product $A \hat{\times} B = A \times \hat{T} \times B$;

⋆ The initiation by Mignani (182) of a *nonpotential-nonunitary scattering theory* reformulated by Santilli (55) as isounitary on iso-Hilbert spaces over isofields, thus recovering causality and propability laws for the first known description of*scattering* among *extended* particles with consequential *contact-nonpotential* inetractions;

⋆ The first and only known exact and invariant representation by Santilli (114,115) of nuclear magnetic moments and other nuclear characteristics thanks to interior mutation of type (3.93), which representation has escaped quantum mechanics for about one century;

⋆ The first and only known model by Animalu (170) and Animalu and Santilli (116) of the Cooper pair in superconductivity with an *attractive force between the two identical electrons* in excellent agreement with experimental data;

⋆ The exact representation via isorelativity by Mignani (118) of the large difference in cosmological redshifts between quasars and galaxies when physically connected;

⋆ The exact representation by Santilli (117) of the internal blue- and red-shift of quasar's cosmological redshift;

⋆ The elimination of the need for a missing mass in the universe by Santilli (34) thanks to isoaxiom V.

Additional important applications of isotopies have been studied by by A. O. E. Animalu, A. K. Aringazin, R. Aslaner, C. Borgi, F. Cardone, J. Dunning-Davies, F. Eder, J. Ellis, J. Fronteau, M. Gasperini, T. L. Gill, J. V. Kadeisvili, A. Kalnay, N. Kamiya, S. Keles, C. N. Ktorides, M. G. Kucherenko, D. B. Lin, C.-X. Jiang, A. Jannussis, R. Mignani, M. R. Molaei, N. E. Mavromatos, H. C. Myung, M. O. Nishioka, D. V. Nanopoulos, S. Okubo, D. L. Rapoport, D. L. Schuch, D. S. Sourlas, A. Tellez-Arenas, Gr. Tsagas N. F. Tsagas, E. Trell, R. Trostel, S. Vacaru,H. E. Wilhelm, W. Zachary, and others. These studies are too numerous to be effectively reviewed in this memoir.

Above all, *hadronic mechanics achieved the main objective for which it was built: the first exact and invariant representation from unadulterated first axiomatic principles of all experimental data of the hydrogen, water and other molecules* by R. M. Santilli and D. D. Shillady (125,126) (see also the comprehensive treatment in monograph (59)). The representation was achieved via the use of nonrelativistic hadronic mechanics based on the simple isounit (3.16) in which, as one can see, *there are no free parameters for ad hoc fits of experimental data,* but only a quantitative description of wave-overlappings, with isorelativistic extension characterized by isounit (3.82).

It should be noted that, whether in valence coupling or not, *electrons repel each other.* Also, the total electric or magnetic forced between neutral atoms are identically null, while exchange, van der Waals and other forces of current use in chemistry are basically insufficient to represent the strength of molecular bonds (59). Studies (125,126) achieved the *first and only known strongly attractive force between pairs of identical electrons in singlet coupling at short distance,* and proved to originate from nonlocal, nonlinear and nonpotential interactions due to deep overlappings of electron's wavepackets in singlet coupling (where the word "strongly" it is not evidently referred to strong interactions). The birth of this new, nonpotential, strongly attractive force between particles in conditions of mutual penetration then implies new structure models of hadrons, nuclei and stars (see below).

As it is well known, the exact representation of molecular data had escaped about one century of attempts via conventional chemistry, because the missing 2% originated precisely from the nonlocal-integral, nonlinear and nonpotential interactions due to deep overlapping of the wavepackets of the valence electrons (Figure 2) which are beyond any descriptive capacity of quantum mechanics.

As it is also well known, improved representations of molecular data have required the "screening of the Coulomb potential," which screening cannot be qualified as characterizing a "quantum" theory since the quantum of energy only exists for the pure Coulomb potential. In any case, screened Coulomb potentials are *nonunitary images* of the Coulomb law, thus being particular cases of the nonunitary/isounitary structure of hadronic mechanics and chemistry (59).

The achievement of a deeper understanding of molecular bonds has far reaching scientific implications. In fact, it confirms that *nonlocal, nonlinear and non potential interactions exist in all interior problems at large, such as the structure of hadrons, nuclei and stars, and imply basically new structure models in which the constituents are isoparticles (irreducible representation of the Poincaré-Santilli isosymmetry), rather than conventional particles in vacuum.*

The original proposal to build hadronic mechanics (38) of 1978 included the proof that *all* characteristics of the π^o meson can be represented in an exact and invariant way via a bound state of one iso-electron $\hat{e}^-$ and its antiparticle $\hat{e}^+$ under conditions of mutual penetration within 10^-13cm,

$$\pi^o = (\hat{e}^-, \hat{e}^+)_{HM}; \tag{3.98}$$

the $\pi^\pm$ meson can be represented via a bound state of three isoelectrons,

$$\pi^\pm = (\hat{e}^-, \hat{e}^\pm, \hat{e}^+)_{HM}; \tag{3.99}$$

and the remaining mesons can be similarly identified as hadronic bound states of massive isoparticles produced free in the spontaneous decays with the lowest mode.

Following the prior achievement of the isotopies of the SU(2) spin (28), Ref. (214) of 1990 achieved for the first time the exact and invariant representation of *all* characteristic of the neutron as a nonrelativistic hadronic bound state of one isoproton and one isoelectron according to Rutherford's original conception of the neutron,

$$n = (\hat{p}^+, \hat{e}^-)_{HM}, \tag{3.100}$$

while the relativistic extension was reached in Ref. (30), jointly with the first isotopies of Dirac's equation. Subsequently, it was easy to see that all unstable baryons can be considered as hadronic bound states of massive isoparticles, again those generally in the spontaneous decays with the lowest modes.

Compatibility of the above new structure models of hadrons with ordinary massive constituents and $SU(3)$-color theories was achieved via the assumption that *quarks are composite,* a view first expressed by Santilli (225) in 1981), and the use of hypermathematics (see Section 5) with *different units for different hadrons* (31). This approach essentially yields the hyperrealization $\hat{SU}(3)$ in which *composite hyperquarks* are characterized by the *multivalued* isounit with isotopic element $\hat{T} = (\hat{T}_u, \hat{T}_d, \hat{T}_s)$, resulting in *hypermultiplets* of mesons, baryons, etc.

The compatibility of this hypermodel with conventional theories is established by the isomorphism between conventional $SU(3)$ and the hyper-$\hat{SU}(3)$, the latter merely being a broader *realization* of the axioms of the former. The significance of this hypermodel is illustrated by the fact that all perturbative series which are divergent for $SU(3)$ are turned into convergent forms because $\hat{T}_u, \hat{T}_d, \hat{T}_s \ll 1$ under which, as indicated earlier, all divergent perturbative series expressed in terms of the conventional product $A \times B$ become convergent when re-expressed in terms of the hytperproduct $A \times \hat{T} \times B$. Compatibility with the structure model of hadrons with ordinary massive constituents is evident from the fact that *quarks result to be composed of ordinary massive isoparticles.*

It should be recalled that none of the above hadronic models are possible for quantum mechanics, e.g., because the representation of the rest energies of hadrons would require "positive binding energies"(since, unlike similar occurrences in nuclear physics, the rest energy of bound states (3.98)-(3.100) is much bigger than the sum of the rest energies of the constituents. Such "positive binding energies are prohibited by quantum mechanics because Schroedinger's equation become inconsistent. These and other objections were resolved by the covering hadronic mechanics due to the isorenormalizations (also called mutations) of the rest energies and other features of the constituents caused by nonlocal, nonlinear and nonpotential interactions.

Predictably, the reduction of the neutron to a bound state of an isoproton and an isoelectron permitted a *new structure model of nuclei as hadronic bound states of isoprotons and isoelectron* (113,114), with the conventional quantum models based on protons and neutron remaining valid in first approximation. The new isonuclear model permitted the first known understanding of the reason why the deuteron ground state has spin 1 since it is a *three-body system* for hadronic mechanics,

$$D = (\hat{p}^+, \hat{e}^-, \hat{p}^+)_{HM}, \tag{3.101}$$

thus admitting 1 as the lowest possible angular momentum, while the ground state of the deuteron quantum mechanics, $D = (p^+, n^o)_{QM}$ should have spin zero since it is a *two body system.* The isonuclear model also permitted the interpretation of other features that had remained unexplained in nuclear physics for about one century such as why the correlation among nucleons is restricted to pairs only.

In particular, the old process of keep adding potentials to the nuclear force without ever achieving an exact representation of nuclear data has been truncated by hadronic mechanics, due to a component of the nuclear force which is nonlocal, nonlinear and nonpotential originating from the mutual penetration of the charge distribution of nucleons established by nuclear data (such as the ratio between nuclear volumes and the sum of the volume of the nucleon constituents).

The reduction of neutron stars and other astrophysical bodies to isoprotons and isoelectrons is then consequential.

3.15: Industrial applications to new clean energies and fuels. In closing, it should be indicated that the studies on isotopies have long passed the level of pure scientific relevance, because they now have direct *industrial applications* fort new clean energies and fuels so much needed by our contemporary society.

As an **illustration at the particle level,** the synthesis of the neutron from one proton and one electron according to Rutherford, Eq. (3.100), has been experimentally confirmed by C. Borghi *et al.* (123) to occur also at low energies, although under a number of conditions studied in monograph (58), and additional tests are under way.

Once Rutherford's original conception of the neutron is rendered acceptable by hadronic mechanics, the electron becomes a physical constituent of the neutron (although in a mutated state). In this case, hadronic mechanics predicts the capability of stimulating the decay of the neutron via photons with suitable resonating frequencies and other means, thus implying the first known form of "hadronic energy" (58) (that is, energy originating in the structure of hadrons, rather than in their nuclear aggregates), which has already been preliminarily confirmed via an experiment conducted by N. Tsagas *et al* (124) (see monograph (58) for scientific aspects and the web site www.betavoltaic.com for industrial profiles).

As an **illustration at the nuclear level,** hadronic mechanics predicts a basically new process for controlled nuclear syntheses which is dramatically different than both the "hot" and the "cold" fusions, and which is currently also under industrial development, which condition prohibits its disclosure in this memoir.

As an **illustration at the molecular level,** the deeper understanding of the structure of *molecules* has permitted the discovery and experimental verifications in Ref.s (27) (see also the studies by Aringazin and his associates in Refs. (128-1130) and monograph (59)) of the new chemical species of *magnecules* consisting of clusters of individual atoms, dimers and molecules under a new bond originating from the electric and magnetic polarization of the orbitals of atomic electrons.

In turn, the new species of magnecules has permitted the industrial synthesis of new fuels without hydrocarbon structure, whose combustion exhaust resolves the environmental problems of fossil fuels by surpassing current exhaust requirement by the U. S. Environmental Protection Agency without catalytic converter or other exhaust purification processes (see monograph (59) for scientific profiles and the web site www.magnegas.com for industrial aspects).

4. CONSTRUCTION OF GENOMECHANICS FROM IRREVERSIBLE PROCESSES

4.1: The scientific unbalance caused by irreversibility. As it is well known, physical, chemical or biological systems are called *irreversible* when their images under time reversal, $t \rightarrow -t$, are prohibited by causality and other laws, as it is the case for nuclear transmutations, chemical reactions and organisms growth. Systems are called *reversible* when their time reversal images are as causal as the original ones, as it is the case for planetary and atomic structures when considered isolated from the rest of the universe (see reprint volume (81) on irreversibility and vast literature quoted therein).

Yet another large scientific unbalance of the 20-th century has been the treatment of *irreversible* systems via mathematical and physical formulations of *reversible* systems which are themselves reversible, resulting in serious limitations in virtually all branches of science. The problem was compounded by the fact that all used formulations were essentially of Hamiltonian type, with the awareness that all known Hamiltonians are reversible (since all known potential interactions are reversible).

This third scientific unbalance was dismissed by academicians with vested interests in reversible theories with unsubstantiated statements, such as "irreversibility is a macroscopic occurrence which disappears when all bodies are reduced to their elementary constituents." The underlying belief is that mathematical and physical theories which are so effective for the study of one electron in a reversible orbit around a proton are tacitly assumed to be equally effective for the study of the same electron when in irreversible motion in the core of a star with the local *nonconservation* of energy, angular momentum, etc. These academic beliefs have been disproved by the following:

THEOREM 4.1 (224): A classical irreversible system cannot be consistently decomposed into a finite number of elementary constituents all in reversible conditions and, vice-versa, a finite collection of elementary constituents all in reversible conditions cannot yield an irreversible macroscopic ensemble.

The occurrence established by the above theorems dismiss all nonscientific conversations which have occurred on irreversibility in the 20-th century, and identify the real scientific needs, the construction of formulations which are *structurally irreversible,* that is, irreversible for all known reversible Hamiltonians, and are applicable at all levels of study, from Newtonian mechanics to second quantization.

4.2: The forgotten legacy of Newton, Lagrange and Hamilton. It should be indicated that the above scientific unbalance existed only in the 20-th century because *Newton's equations (1) are generally irreversible* since, as recalled in the preceding section, Newton's force $F(t, x, v)$ can be decomposed into the sum of variationally selfadjoint and nonselfadjoint components (48,51)

$$m_\alpha \times a_{k\alpha} = F^{SA}_{k\alpha} + F^{NSA}_{k\alpha}, \tag{4.1a}$$

$$F^{SA} = -\partial V/\partial x, F^{NSA} \neq -\partial V/\partial x, k = 1, 2, 3, \alpha = 1, 2, ..., n. \tag{4.1b}$$

It is evident that, since all known F^{SA} are reversible, *in Newtonian mechanics irreversibility originates in the contact nonpotential forces* F^{NSA}.

In a way fully aligned with Newton's teaching, Lagrange (2) and Hamilton (3) formulated their celebrated analytic equations in terms of a function, today called the Lagrangian $L(x, v)$ and the Hamiltonian $H(x, p)$, representing F^{SA}, plus *external terms* representing precisely the contact nonpotential forces F^{NSA},

$$\frac{d}{dt}\frac{\partial L(x, v)}{\partial v} - \frac{\partial L(x, v)}{\partial x} = F^{NSA}(t, x, v), \tag{4.2a}$$

$$\frac{dx}{dt} - \frac{\partial H(x, p)}{\partial p} = 0, \frac{dp}{dt} + \frac{\partial H(x, p)}{\partial x} = F^{NSA}(t, x, p), \tag{4.2b}$$

with time evolution for an observable $A(x, p)$ in phase space over R characterized by the brackets

$$\frac{dA}{dt} = (A, H, F^{NSA}) =$$

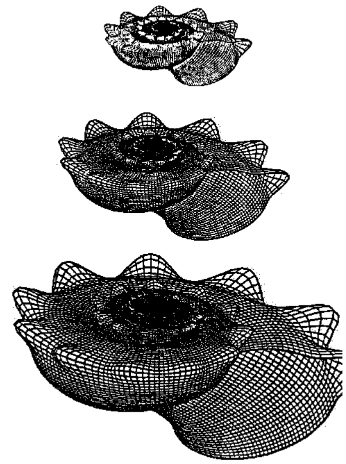

Figure 3: An illustration via sea shells growth of the third scientific unbalance of the 20-th century, the lack of a structurally irreversible mathematics (that is, a mathematics whose basic axioms are not invariant under time reversal) for quantitative representations of irreversible processes. The unbalance is due to the fact that all formulations used until now are of Hamiltonian type, while all known Hamiltonians and their background mathematics are reversible, thus implying the study of irreversible systems via fully reversible formulations.

$$= (\frac{\partial A}{\partial x^k_\alpha} \times \frac{\partial H}{\partial p_{k\alpha}} - \frac{\partial A}{\partial p_{k\alpha}} \times \frac{\partial H}{\partial x^k_\alpha}) + \frac{\partial A}{\partial p_{k\alpha}} \times F^{NSA}_{k\alpha}. \quad (4.3)$$

Since all known Lagrangians and Hamiltonians are reversible in time, *according to the teaching of Lagrange and Hamilton, irreversibility is characterized, again, by the external terms representing contact zero-range interactions among extended particles.*

At the beginning of the 20-th century, Lagrange's and Hamilton's external terms were truncated, resulting in analytic equations

$$\frac{d}{dt}\frac{\partial L(x,v)}{\partial v} - \frac{\partial L(x,v)}{\partial x} = 0, \quad (4.4a)$$

$$\frac{dx}{dt} - \frac{\partial H(x,p)}{\partial p} = 0, \frac{dp}{dt} + \frac{\partial H(x,p)}{\partial x} = 0, \quad (4.4b)$$

with time evolution characterized by the familiar Lie brackets

$$\frac{dA}{dt} = [A, H] =$$

$$= \frac{\partial A}{\partial x^k_\alpha} \times \frac{\partial H}{\partial p_{k\alpha}} - \frac{\partial A}{\partial p_{k\alpha}} \times \frac{\partial H}{\partial x^k_\alpha}, \quad (4.5)$$

which are fully reversible.

The above occurrence was due to the successes of the truncated analytic equations for the representation of planetary and atomic structures, resulting in their use for virtually all scientific inquiries of the 20-th century. In turn, the assumption of the truncated analytic equations as the ultimate formulation of science implied the scientific unbalance under consideration here because planetary and atomic structures are fully reversible, thus lacking sufficient generalities for all of nature.

4.3: Catastrophic inconsistencies of formulations with external terms. More recent studies (23,38) have shown that the true Lagrange's and Hamilton's equations (those with external terms) cannot be used in applications due to a number of insufficiencies, such as:

(1) the lack of invariant numerical predictions in accordance with Theorem 3.1 (due to their evident noncanonical character);

(2) the lack of characterization of *any* algebra by the brackets of the time evolution, let alone the loss of all Lie algebras, because brackets (A, H, F^{NSA}) of Eqs. (4.3) violate the right distributive and scalar laws as necessary to characterize an algebra commonly understood in contemporary mathematics since they are *triple systems*);

(3) the lack of a topology suitable to represent contact nonpotential interactions among extended particles since the topology of conventional Hamiltonian formulation is strictly local-differential, thus solely characterizing point particles; and other limitations.

The *only* resolution of these problematic aspects known to this author was the construction of the novel structurally irreversible mathematics indicated earlier. Stated in different terms, the manifestly inconsistent reduction of irreversible macroscopic systems to elementary particles in reversible conditions was due, again, to *insufficiencies of the used mathematics.*

It should be noted that the isomathematics of the preceding section is also reversible in time because the isounit is Hermitean, thus lacking the mathematical characterization of time reversal, and confirming the need of constructing of a broader mathematics specifically suited to represent irreversibility.

4.4: Initial versions of irreversible mathematics. The achievement of a structurally irreversible mathematics resulted to be a long scientific journey due to the need of achieving *invariance under irreversible conditions.* The first studies can be traced back to Ref. (8) of 1967 which presented the first known *parametric deformation of Lie algebras* with product

$$
\begin{aligned}
(A, B) &= p \times A \times B - q \times B \times A = \\
&= v \times (A \times B - B \times A) + w \times (A \times B + B \times A) = \\
&= v \times [A, B] + w \times \{A, B\}, \qquad (4.6)
\end{aligned}
$$

where p, q, and $p \pm q$ are non-null parameters, $v = p + q, w = q - p$, and A, B are Hermitean matrices.

The studies continued with the first known presentation in Ref. (38) of the *operator deformations of Lie algebra* with product

$$(A\hat{,}B) = A \times P \times B - B \times Q \times A =$$

$$= (A \times T \times B - B \times T \times A) + (A \times W \times B + B \times W \times A) =$$

$$= [A\hat{,}B] + \{A\hat{,}B\}, T + W = P, W - T = Q, \tag{4.7}$$

where P, Q and $P \pm Q$ are nowhere singular matrices.

On historical grounds, the above deformations were introduced in Refs. (8,38) as realizations of *Albert's Lie-admissible and Jordan-admissible products* (7), namely, products whose antisymmetric and symmetric parts are Lie and Jordan, respectively,

$$(A\hat{,}B) - (B\hat{,}B) = 2 \times [A\hat{,}B] = Lie, \tag{4.8a}$$

$$(A\hat{,}B) + (B\hat{,}A) = 2 \times (A\hat{,}B) = Jordan. \tag{4.8b}$$

Note, however, that the Lie and Jordan algebras attached to brackets $(A\hat{,}B)$ are not conventional because of their broader *isotopic* nature [4].

The transition from the parameter to the operator deformations of Lie algebras was mandatory because all time evolution which can be characterized by the former brackets are *nonunitary.* Therefore, the reader can easily verify that the application of a nonunitary transform to the parametric deformations leads to the operator ones,

$$i \times dA/dt = (A, B), A(t) = U \times A(0) \times U^\dagger, U \times U^\dagger \neq I, \tag{4.9a}$$

$$U \times (A, B) \times U^\dagger = (A'\hat{,}B')', \tag{4.9b}$$

$$A' = U \times A \times U^\dagger, B' = U \times B \times U\dagger, \tag{4.9c}$$

$$T = v \times (U \times U^\dagger)^{-1}, W = w \times (U \times U^\dagger)^{-1}. \tag{4.9d}$$

Operator deformations (4.7) (rather than the parametric deformations (4.6)) are promising for the representation of irreversibility because they are no longer totally anti-symmetric (as it is the case for Lie brackets) and, therefore, they can indeed represent *nonconservation* as needed in irreversible processes. Moreover, operator deformations (4.7) are *universal* in the sense of admitting as particular cases all infinitely possible algebras as currently known in mathematics (those characterized by a bilinear product), including Lie, Jordan, Kac-Moody, supersymmetric and all other possible algebras. Finally, the joint Lie- and Jordan-admissibility is preserved by any additional nonunitary transforms,

$$Z \times (A\hat{,}B) \times Z^\dagger = A' \times P' \times B' - B' \times Q' \times A' = (A'\hat{,}\hat{B})', \tag{4.10a}$$

$$Z \times Z^\dagger \neq I, A' = Z \times A \times Z^\dagger, B' = Z \times B \times Z^\dagger \tag{4.10b}$$

$$P' = Z^{\dagger -1} \times P \times Z^{-1}, Q' = Z^{\dagger -1} \times Q \times Z^{-1}, \tag{4.10b}$$

thus confirming that brackets $(A\hat{,}B)$ characterize the most general possible algebras.

Nevertheless, all parametric and operator deformations are afflicted by the catastrophic mathematical and physical inconsistencies of Theorem 3.1 because of the lack of invariance of the deformation parameters $P \rightarrow P' \neq P, Q \rightarrow Q' \neq Q$ (or, equivalently, of the product).

During the last two decades of the 20-th century, the physical and mathematical literature saw an explosion of contributions in Lie deformations which continues to this day, although generally without a quotation of their origination in Refs. (8,23,38), without a quotation of their Lie- and Jordan-admissible content (7), and, above all, without a quotation of the rather vast literature on their catastrophic inconsistencies (see, Refs. (171-175) and memoir (46) and literature quoted therein). By contrast, by the mid 1980's this author had abandoned the study of Lie deformations according to their original formulations (8,23,38) because of said catastrophic inconsistencies.

4.5: Elements of genomathematics. A breakthrough occurred with the discovery, apparently done for the first time by R. M. Santilli in Ref. (12) of 1993, that *the axioms of a field also hold when the ordinary product of numbers $a \times b$ is ordered to the right, $a > b$, or, separately, ordered to the left, $a < b$.* In turn, such an order permitted the construction of *two* generalized units, called *genounits to the right and to the left*

$$I = Diag.(1,1,...,1) \rightarrow \hat{I}^{>}(t,x,v,\psi,\partial_x\psi,...) = 1/\hat{T}^{>}(t,x,v,\psi,\partial_x\psi,...) > 0, \quad (4.11a)$$

$$I = Diag.(1,1,...,1) \rightarrow <> \hat{I}(t,x,v,\psi,\partial_x\psi,...) = 1/{}^{<}\hat{T}(t,x,v,\psi,\partial_x\psi,...), \quad (4.11b)$$

$$\hat{I}^{>} = ({}^{<}\hat{I})^{\dagger}, \quad (4.11c)$$

with corresponding *ordered genoproducts to the right and to the left*

$$A \times B \rightarrow A > B = A \times \hat{T}^{>} \times B, \quad (4.12a)$$

$$A \times B \rightarrow\rightarrow A < B = A \times^{<} T \times B, \quad (4.12b)$$

$$A > B = (B < A)^{\dagger}, \quad (4.12c)$$

under which the left and rights character of the genounits is preserved,

$$I \times A = A \times I = A \rightarrow \hat{I}^{>} > A = A > \hat{I}^{>} = A, \quad (4.13a)$$

$$I \times A = A \times I = A \rightarrow^{<} \hat{I} < A = A <^{<} \hat{I} = A, \quad (4.13b).$$

for all (Hermitean elements A, B, of the considered set. Examples of genounits and genoproducts will be provided shortly.

In this way, the ordering ">" can describe motion forward in time while the ordering "<" can describe motion backward in time, with interconnecting Hermitean (or transposed) conjugation. This approach permitted the embedding of irreversibility in the most fundamental quantities, the basic units and operations, thus assuring *ab initio* the construction of a structurally irreversible mathematics, today known as *genomathematics,* as summarized below.

DEFINITION 4.1: Let $F = F(a, +, \times)$ be a field as per Definition 2.1. The *forward genofields* (first introduced in Ref. (12) of 1993) are rings $\hat{F}^> = \hat{F}^>(\hat{a}^>, \hat{+}^>, >)$ with *forward genonumbers*

$$\hat{a}^> = a \times \hat{I}^>, \tag{4.14}$$

associative, distributive and commutative *forward genosum*

$$\hat{a}^> \hat{+}^> \hat{b}^> = (a + b) \times \hat{I}^> = \hat{c}^> \in \hat{F}^>, \tag{4.15}$$

associative and distributive *forward genoproduct*

$$\hat{a}^> > \hat{b}^> = \hat{a} > \times \hat{T}^> \times \hat{b}^> = \hat{c}^> \in \hat{F}, \tag{4.16}$$

additive forward genounit

$$\hat{0}^> = 0, \hat{a}^> \hat{+}^> \hat{0}^> = \hat{0}^> \hat{+}^> \hat{a}^> = \hat{a}^> \in \hat{F}^>, \tag{4.17}$$

and *multiplicative forward genounit*

$$\hat{I}^> = 1/\hat{T}^>, \hat{a}^> > \hat{I}^> = \hat{I}^> > \hat{a}^> = \hat{a}^> \in \hat{F}^>, \forall \hat{a}^>, \hat{b}^> \in \hat{F}^>, \tag{4.18}$$

where $\hat{I}^>$ is a complex-valued non-Hermitean, or real-value non-symmetric, everywhere invertible quantity generally *outside* F. The *backward genofields* $^<\hat{F}(^<\hat{a},^< \hat{+}, <)$, their elements, units and their operations are given by the Hermitean conjugate (or transposed) of the corresponding quantities and their operations in $\hat{F}^>(\hat{a}^>, \hat{+}^>, \hat{\times}^>)$, e.g.,

$$^<\hat{I} = (\hat{I}^>)^\dagger, etc. \tag{4.19}$$

LEMMA 4.1: Forward and backward genofields are fields with characteristic zero (namely, they verify all axioms of said fields).

In Sect. 2 we pointed out that the conventional product "2 multiplied by 3" is not necessarily equal to 6 because, depending on the assumed unit and related product, it can be -6. In Section 3 we pointed out that the same product "2 multiplied by 3" is not necessarily equal to $+6$ or -6, because it can also be equal to an arbitrary number, or a matrix or an an integrodifferential operator. In this section we point out that "2 multiplied by 3" can be ordered to the right or to the left, and yield different numerical results for different orderings, "$2 > 3 \neq 2 < 3$, all this by continuing to verify the axioms of a field per each order (12).

Once the forward and backward fields have been identified, the various branches of genomathematics can be constructed via simple compatibility arguments, resulting in the *genofunctional analysis, genodifferential calculus,* etc (14,54,55). We have in this way the *genodifferentials and genoderivatives*

$$\hat{d}^> x = \hat{T}^>_x \times dx, \frac{\hat{\partial}^>}{\hat{\partial}^> x} = \hat{I}^>_x \times \frac{\partial}{\partial x}, etc. \tag{4.20}$$

Particularly intriguing are the *genogeometries* (*loc. cit.*) because they admit *nonsymmetric metrics,* such as the *genoriemannian metrics*

$$g^{>}(x) = \hat{T}^{>}(x) \times \eta, \tag{4.21}$$

where η is the Minkowski metric and $\hat{T}^{>}(x)$ is a real-values, nowhere singular, 4×4 *nonsymmetric* matrix, while bypassing known inconsistencies since they are referred to the *nonsymmetric genounit*

$$\hat{I}^{>} = 1/\hat{T}^{>}. \tag{4.22}$$

In this way, *genogeometries are structurally irreversible* and actually represent irreversibility with their most central geometric notion, the metric.

4.6: Lie-Santilli genotheory and its isodual. Particularly important for this note is the lifting of Lie's theory permitted by genomathematics, first identified by R. M. Santilli in Ref. (23) of 1978, and today knows as the *Lie-Santilli genotheory* [7,8], which is characterized by:

(1) The *forward and backward universal enveloping genoassociative algebra* $\hat{\xi}^{>}, {}^{<}\hat{\xi}$, with infinite-dimensional basis characterizing the*Poincaré-Birkhoff-Witt-Santilli genotheorem*

$$\hat{\xi}^{>} : \hat{I}, \hat{X}_i, \hat{X}_i > \hat{X}_j, \hat{X}_i > \hat{X}_j > \hat{X}_k, ..., i \leq j \leq k, \tag{4.23a}$$

$${}^{<}\hat{\xi} : \hat{I}, \hat{X}_i, \hat{X}_i < \hat{X}_j, \hat{X}_i < \hat{X}_j < \hat{X}_k, ..., i \leq j \leq k; \tag{4.23b}$$

where the "hat" on the generators denotes their formulation on genospaces over genofields and their Hermiticity implies that $\hat{X}^{>} = {}^{<}\hat{X} = \hat{X}$;

(2) The *Lie-Santilli genoalgebras* characterized by the universal, jointly Lie-and Jordan-admissible brackets (4.7),

$${}^{<}\hat{L}^{>} : (\hat{X}_i \hat{,} \hat{X}_j) = \hat{X}_i < \hat{X}_j - \hat{X}_j > \hat{X}_i = \hat{C}_{ij}^{k} \hat{\times} \hat{X}_k, \tag{4.24}$$

although now formulated in an invariant form (see below);

(3) The *Lie-Santilli genotransformation groups*

$$ {}^{<}\hat{G}^{>} : \hat{A}(\hat{w}) = (\hat{e}_{>}^{\hat{i} \hat{\times} \hat{X} \hat{\times} \hat{w}}) > \hat{A}(\hat{0}) < ({}_{<}\hat{e}^{-\hat{i} \hat{\times} \hat{w} \hat{\times} \hat{X}}) =$$

$$= (e^{i \times \hat{X} \times \hat{T}^{>} \times w}) \times A(0) \times (e^{-i \times w \times {}^{<}\hat{T} \times \hat{X}}), \tag{4.25}$$

where $\hat{w}^{>} \in \hat{R}^{>}$ are the *genoparameters;* the *genorepresentation theory,* etc.

The mathematical implications of the Lie-Santilli genotheory are significant because of the admission as particular cases of all possible algebras, as well as because, when computed on the *genobimodule* ${}^{<}\hat{\xi} \times \hat{\xi}^{>}$ *Lie-admissible algebras verify all Lie axioms,* while deviations from Lie algebras emerge only in their *projection* on the bimodule ${}^{<}\xi \times \xi^{>}$ of the conventional Lie theory. This is due to the fact that the computation of the left action $A < B = A \times^{<} \hat{T} \times B$ on ${}^{<}\hat{\xi}$ (that is, with respect to the genounit ${}^{<}\hat{I} = 1/{}^{<}\hat{T}$) yields the save value as the computation of the conventional product $A \times B$ on ${}^{<}\xi$ (that is, with respect to the trivial unit I), and the same occurs for the value of $A > B$ on $\hat{\xi}^>$. In this way, thanks to genomathematics, *Lie algebras acquire a towering significance in view of the possibility of reducing all known algebras to primitive Lie axioms.*

The physical implications of the Lie-Santilli genotheory ar equally significant. In fact, Noether's theorem on the reduction of conservation laws to primitive Lie symmetries can be generalized to the *reduction of, this time, nonconservation laws to primitive Lie-Santilli genosymmetries.* As a matter of fact, this reduction was the very first motivation that suggested the construction of the genotheory in memoir (23) (see also monographs (49,50)). The reader can then foresee similar liftings of all remaining physical aspects treated via Lie algebras.

The construction of the isodual Lie-Santilli;i genotheory is an instructive exercise for readers interested in learning the new methods.

The physical theories characterized by genomathematics can be summarized as follows.

4.7: Geno-Newtonian Mechanics and its isodual. Recall that, for the case of isotopies, the basic Newtonian systems are given by those admitting nonconservative internal forces restricted by certain constraints which verify total conservation laws (closed non-Hamiltonian systems). For the case of the genotopies under consideration here, the basic Newtonian systems are the conventional nonconservative systems (4.1) without subsidiary constraints (open non-Hamiltonian systems). In this case irreversibility is characterized by nonselfadjoint forces, as indicated earlier.

The *forward geno-Newtonian mechanics and its isodual* is a generalization of Newtonian mechanics for the description of motion forward in time of the latter systems via a structurally irreversible mathematics. The new mechanics is characterized by (14): the *forward genotime* $\hat{t}^> = t \times \hat{I}_t^>$ with (nowhere singular and non-Hermitean) *forward time genounit* $\hat{I}_t^> = 1/\hat{T}_t^> \neq \hat{I}_t^{>\dagger}$, related *forward time genospace* $\hat{S}_t^>$ over the *forward time genofield* $\hat{R}_t^>$; the *forward genocoordinates* $\hat{x}^> = x \times \hat{I}_x^>$ with (nowhere singular non-Hermitean) *forward coordinate genounit* $\hat{I}_x^> = 1/\hat{T}_x^> \neq \hat{I}_x^{>\dagger}$ with *forward coordinate genospace* $\hat{S}_x^>$ and related *forward coordinate genofield* $\hat{R}_x^>$; and the *forward genospeeds* $\hat{v}^> = \hat{d}^>\hat{x}^>/\hat{d}^>\hat{t}^>$ with (nowhere singular and non-Hermitean) *forward speed genounit* $\hat{I}_v^> = 1/\hat{T}_v^> \neq \hat{I}_v^{>\dagger}$ with related *forward speed genospace* $\hat{S}_x^>$ and *forward speed genofield* $\hat{R}_v^>$. Note that, to verify the condition of non-Hermiticity, the time genounits should be at least *complex valued,* and the same then occurs for the other genounits.

The representation space is then given by the Kronecker product

$$\hat{S}_{Tot}^> = \hat{S}_t^> \times \hat{S}_x^> \times \hat{S}_v^>, \tag{4.26}$$

defined over the genofield

$$\hat{R}_{tot}^> = \hat{R}_t^> \times \hat{R}_x^> \times \hat{R}_v^>, \tag{4.27}$$

with total genounit

$$\hat{I}_{tot}^> = \hat{I}_t^> \times hatI_x^> \times \hat{I}_v^>. \tag{4.28}$$

The basic equations are given by the *forward geno-Newton equations,* also known as *Newton-Santilli genoequations*, first proposed in memoir (14) via the genodifferential calculus, also known as *forward Newton-Santilli genoequations* [8-11]

$$\hat{m}^{>} >_{\alpha} \frac{\hat{d}^{>}\hat{v}^{>}_{k\alpha}}{\hat{d}^{>}\hat{t}^{>}} = -\frac{\hat{\partial}^{>}\hat{V}^{>}}{\hat{\partial}^{>}\hat{x}^{>k}_{\alpha}}. \tag{4.29}$$

The *backward geno-Newton equations* is characterized by backward genounits can be obtained via transpose conjugation of the forward formulation.

As one can see, the representation of Newton's equations is done in a way similar to the isotopic case, the main difference being that the basic unit is now no longer symmetric. Note that in Newton's equations the nonpotential forces are part of the applied force F, while in the geno-Newton equations nonpotential forces are represented by the forward genounits, or, equivalently, by the forward genodifferential calculus, in a way essentially similar to the case of isotopies. The main difference is that isounits are Hermitean, thus implying the equivalence of forward and backward motions, while genounits are non-Hermitean, thus implying irreversibility.

Note also that the topology underlying Newton's equations is the conventional, Euclidean, local-differential topology which, as such, can only represent point particles. By contrast, the topology underlying the geno-Newton equations is the *Santilli-Sourlas-Tsagas genotopology* (14,139) for the representation of extended, nonspherical and deformable particles via forward genounits, e.g., of the diagonal type

$$\hat{I}^{>} = Diag.(n_1^2, n_2^2, n_3^2, n_4^2) \times \Gamma^{>}(t, x, v, ...), \tag{4.30}$$

where $n_k^2, k = 1, 2, 3$ represents the semiaxes of an ellipsoid, n_4^2 represents the density of the medium in which motion occurs (with more general nondiagonal realizations here omitted for simplicity), and $\Gamma^{>}$ represents contact interactions occurring for the motion forward in time.

The construction of the isodual image of the above geno-Newtonian mechanics is instructive to understand the difference between isoduality and motion backward in time.

4.8: Geno-Hamiltonian mechanics and its isodual. The most effective setting to introduce real-valued and non-Hermitean (thus non-symmetric) genounits is in the 6n-dimensional *forward genocotangent bundle* (geno-phase-space) with local genocoordinates and their conjugate

$$\hat{a}^{>\mu} = a^{\rho} \times \hat{I}^{>\mu}_{1\rho}, (\hat{a}^{>\mu}) = \begin{pmatrix} \hat{x}^{>k}_{\alpha} \\ \hat{p}^{>}_{k\alpha} \end{pmatrix}, \hat{R}^{>}_{\mu} = R_{\rho} \times \hat{I}^{>\rho}_{2\mu}, (\hat{R}^{>}_{\mu}) = (\hat{p}_{k\alpha}, \hat{0}), \qquad (4.31a),$$

$$\hat{I}^{>}_1 = 1/\hat{T}^{>}_1 = (\hat{I}^{>}_2)^T = (1/\hat{T}^{>}_2)^T, k = 1, 2, 3, \alpha = 1, 2, ..., n, \mu, \rho = 1, 2, ...6n, \qquad (4.31a)$$

where the superscript T stands for transposed, with nowhere singular, real-valued and non-symmetric genometric and related invariant

$$\hat{\delta}^{>} = \delta_{6n\times 6n} \times \hat{T}^{>}_{1\ 6n\times 6n}, \tag{4.32a}$$

$$\hat{a}^{>\mu} > \hat{R}^{>}_{\mu} = \hat{a}^{>\rho} \times \hat{T}^{>\beta}_{1\rho} \times \hat{R}^{>}_{\beta} = a^{\rho} \times \hat{I}^{>\beta}_{2\rho} \times R_{\beta}. \tag{4.32b}$$

In this case we have the following *genoactionprinciple* (14)

$$\hat{\delta}^{>}\hat{\mathcal{A}}^{>} = \hat{\delta}^{>}\hat{\int}^{>} [\hat{R}^{>}_{\mu} >_a \hat{d}^{>}\hat{a}^{>} - \hat{H}^{>} >_t \hat{d}^{>}\hat{t}^{>}] =$$

$$= \delta\int [R_\mu \times \hat{T}^{>\mu}_{1\nu}(t,x,p,...) \times d(a^\beta \times \hat{I}^{>\nu}_{1\beta}) - H \times dt] = 0, \qquad (4.33)$$

where the second expression is the projection on conventional spaces over conventional fields and we have assumed for simplicity that the time genounit is 1.

It is easy to prove that the above genoprinciple characterizes the following *forward geno-Hamilton equations,* also called *forward Hamilton-Santilli genoequations* (originally proposed in Ref. (23) of 1978 with conventional mathematics and in ref. (14) of 1996 with genomathematics; see also Refs. (28,51,52,55))

$$\hat{\omega}_{\mu\nu}\hat{\times}\frac{\hat{d}\hat{a}^\nu}{\hat{d}\hat{t}} - \frac{\hat{\partial}\hat{H}(\hat{a})}{\hat{\partial}\hat{a}^\mu} =$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} dx/dt \\ dp/dt \end{pmatrix} - \begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \partial H/\partial x \\ \partial H/\partial p \end{pmatrix} = 0, \qquad (4.34a)$$

$$\hat{\omega} = (\frac{\hat{\partial}R_\nu}{\hat{\partial}\hat{a}^\mu} - \frac{\hat{\partial}\hat{R}_\mu}{\hat{\partial}\hat{a}^\nu}) \times \hat{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \hat{I}, \qquad (4.34b)$$

$$K = F^{NSA}/(\partial H/\partial p). \qquad (4.34c)$$

The time evolution of a quantity $\hat{A}^{>}(\hat{a}^{>})$ on the forward geno-phase-space can be written in terms of the following brackets

$$\frac{d\hat{A}^{>}}{dt} = (\hat{A}^{>}, \hat{H}^{>}) = \frac{\hat{\partial}^{>}\hat{A}^{>}}{\hat{\partial}^{>}\hat{a}^{>\mu}}\hat{\times}\hat{\omega}^{\mu\nu}\hat{\times}\frac{\hat{\partial}^{>}\hat{H}^{>}}{\hat{\partial}\hat{a}^{>\nu}} =$$

$$= \frac{\partial\hat{A}^{>}}{\partial\hat{a}^{>\mu}} \times S^{\mu\nu} \times \frac{artoa;\hat{H}^{>}}{\partial\hat{a}^{>\nu}} =$$

$$= (\frac{\partial\hat{A}^{>}}{\partial\hat{x}^{>k}_\alpha} \times \frac{\partial\hat{H}^{>}}{\partial\hat{p}^{>}_{k\alpha}} - \frac{\partial\hat{A}^{>}}{\partial\hat{p}^{>}_{k\alpha}} \times \frac{\partial\hat{H}^{>}}{\partial\hat{x}^{>k}_\alpha}) + \frac{\partial\hat{A}^{>}}{\partial\hat{p}^{>}_{k\alpha}} \times K^k_k \times \frac{\partial\hat{H}^{>}}{\partial\hat{p}^{>}_{k\alpha}}. \qquad (4.35a)$$

$$S^{>\mu\nu} = \omega^{\mu\rho} \times \hat{I}^{2\nu}_\rho, \omega^{\mu\nu} = (||\omega_{\alpha\beta}||^{-1})^{\mu\nu}, \qquad (4.35b)$$

where $\omega^{\mu\nu}$ is the conventional Lie tensor and, consequently, $S^{\mu\nu}$ is Lie-admissible in the sense of Albert (7).

As one can see, the important consequence of genomathematics and its genodifferential calculus is that of turning the triple system (A, H, F^{NSA}) of Eqs. (4.3) in the bilinear form $(A\hat{;}B)$ of brackets (4.35a),m thus regaining the existence of a consistent algebra in the brackets of the time evolution, for which central purpose genomathematics was built (since the multiplicative factors represented by K are fixed for each given system). The invariance of such a formulation will be proved shortly.

It is easy to verify that the above identical reformulation of Hamilton's historical time evolution (4.3) correctly recovers the *time rate of variations of physical quantities* in general, and that of the energy in particular,

$$\frac{dA}{dt} = [\hat{A}^>, \hat{H}^>] + \frac{\partial \hat{A}^>}{\partial \hat{p}^>_{k\alpha}} \times F^{NSA}_{k\alpha}. \tag{4.36a}$$

$$\frac{dH}{dt} = [\hat{H}^>, \hat{H}^>] + \frac{\partial \hat{H}^>}{\partial \hat{p}^>_{k\alpha}} \times F^{NSA}_{k\alpha} = v^k_\alpha \times F^{NSA}_{k\alpha}. \tag{4.36b}$$

It is easy to show that genoaction principle (4.33) characterizes the following *Hamilton-Jacobi-Santilli genoequations*

$$\frac{\hat{\partial}^> \mathcal{A}^>}{\hat{\partial}^> \hat{t}^>} + \hat{H}^> = 0, \tag{4.37a}$$

$$(\frac{\hat{\partial}^> \mathcal{A}^>}{\hat{\partial}^> \hat{a}^{>\mu}}) = (\frac{\hat{\partial}^> \mathcal{A}^>}{\hat{\partial}^> x^{>k}_\alpha}, \frac{\hat{\partial}^> \mathcal{A}^>}{\hat{\partial}^> p^>_{k\alpha}}) = (\hat{R}^>_\mu) = (\hat{p}^>_{k\alpha}, \hat{0}), \tag{4.37b}$$

which confirm the property (crucial for genoquantization as shown below) that the genoaction is indeed independent of the linear momentum.

Note the *direct universality* of Eqs. (4.33) for the representation of all infinitely possible Newton equations (4.1) (universality) directly in the fixed frame of the experimenter (direct universality). Note also that, at the abstract, realization-free level, Geno-Hamilton equations (4.34) *coincide* with Hamilton's equations *without* external terms, yet represent those *with* external terms. The latter are reformulated via genomathematics as the only known way to achieve invariance while admitting a consistent algebra in the brackets of the time evolution (38). Therefore, genohamilton equations (4.34) are indeed irreversible for all possible reversible Hamiltonians, as desired. The origin of irreversibility rests in the contact nonpotential forces according to Lagrange's and Hamilton's teaching. Note finally that the extension of Eqs. (4.9) to include nontrivial genotimes implies a major broadening of the theory we cannot review for brevity (14,55).

The above geno-Hamiltonian mechanics requires, for completness, *three* additional formulations, the *backward geno-Hamiltonian mechanics* for the description of *matter moving backward in time,* and the isoduals of both the forward and backward mechanics for the description of *antimatter.* The construction of these additional mechanics is lefty to the interested reader.

4.9: Genotopic Branch of Hadronic Mechanics and its isodual. A simple genotopy of the naive or symplectic quantization applied to Eqs. (4.37) yields the *genotopic branch of hadronic mechanics* defined on the *forward genotopic Hilbert space* $\hat{\mathcal{H}}^>$ with *forward genoinner product* $< \hat{\psi}| > |\hat{\psi} > \times \hat{I}^> \in \hat{C}^>$. The resulting genotopy of quantum mechanics is characterized by the *forward geno-Schroedinger equations* (first formulated in Refs. (42,179) via conventional mathematics and in Ref. (14) via genomathematics)

$$\hat{i}^> > \frac{\hat{\partial}^>}{\hat{\partial}^> \hat{t}^>} |\hat{\psi}^> >= \hat{H}^> > |\hat{\psi}^> >=$$

$$= \hat{H}(\hat{x}, \hat{v}) \times \hat{T}^>(\hat{t}^>, \hat{x}^>, \hat{p}^>, \hat{\psi}^>, \hat{\partial}^> \hat{\psi}^> ....) \times |\hat{\psi}^> >= E^> > |\psi^> >, \quad (4.38a)$$

$$\hat{p}^>_k \hat{\times} |\hat{\psi}^> >= -\hat{i}^> > \hat{\partial}^>_k |\hat{\psi}^> >= -i \times \hat{I}^{>i}_k \times \partial_i |\hat{\psi}^> >,$$

$$\hat{I}^> > |\hat{\psi}^> >= |\hat{\psi}^> >, \quad (4.38b)$$

with conjugate backward equations obtained via Hermitean conjugation.

Note the crucial independence of isoaction $\hat{\mathcal{A}}^>$ in principle (4.33) from the linear momentum, as expressed by the Hamilton-Jacobi-Santilli genoequations (4.37). In fact, such independence assures that genoquantization yields a genowavefunction solely dependent on time and coordinates, $\hat{\psi}^> = \hat{\psi}^>(t, x)$. Other geno-Hamiltonian mechanics do not verify such a condition, thus implying genowavefunctions with an explicit dependence also on linear momenta, $\hat{\psi}^> = \hat{\psi}^>(t, x, p)$ which violate the abstract identity of quantum and hadronic mechanics andwhose treatment in any case is beyond our operator knowledge at this writing.

The complementary *geno-Heisenberg equations* are given by in their finite and infinitesimal forms (first formulated in Ref. (38) via conventional mathematics and in Ref. (14) via genomathematics)

$$\hat{A}(\hat{t}) = (\hat{e}_>^{\hat{i} \hat{\times} \hat{H} \hat{\times} \hat{t}}) > \hat{A}(\hat{0}) < ({}_<\hat{e}^{-\hat{i} \hat{\times} \hat{t} \hat{\times} \hat{H}}) =$$

$$= (e^{i \times \hat{H} \times \hat{T}^> \times t}) \times A(0) \times (e^{-i \times t \times {}^<\hat{T} \times \hat{H}}), \quad (4.39a)$$

$$\hat{i} \hat{\times} \frac{\hat{d}\hat{A}}{\hat{d}\hat{t}} = (\hat{A} \hat{,} \hat{H}) = \hat{A} < \hat{H} - \hat{H} > \hat{A} =$$

$$= \hat{A} \times^< \hat{T}(\hat{t}, \hat{x}, \hat{p}, \hat{\psi}, ...) \times \hat{H} - \hat{H} \times \hat{T}^>(\hat{t}, \hat{x}, \hat{p}, \hat{\psi}, ...) \times \hat{A}, \quad (4.39b)$$

$$(\hat{x}^i \hat{,} \hat{p}_j) = i \times \delta^i_j \times \hat{I}^>, (\hat{x}^i \hat{,} \hat{x}^j) = (\hat{p}_i \hat{,} \hat{p}_j) = 0, \quad (4.39c)$$

where time has no arrow, since Heisenberg's equations are computed at a fixed time.

The *genoexpectation values* of an observable for the forward motion $\hat{A}^>$ are then given by

$$\frac{< \hat{\psi}| > \hat{A}^> > |\hat{\psi} >}{< \hat{\psi}| > |\hat{\psi} >} \times \hat{I}^> \in \hat{C}^> \quad (4.40)$$

In particular, the genoexpectation values of the genounit recover the conventional Planck's unit,

$$\frac{< \hat{\psi}| > \hat{I}^> > |\hat{\psi} >}{< \hat{\psi}| > |\hat{\psi} >} = I, \quad (4.41)$$

thus confirming that the genotopies are "hidden" in the abstract axioms of quantum mechanics much along the celebrated Einstein-Podolsky-Rosen argument.

Note that *geno-Hermiticity coincides with conventional Hermiticity.* As a result, all quantities which are observable for quantum mechanics remain observable for the above formulation. However, unlike quantum mechanics, physical quantities are generally *non-conserved,* as it must be the case for the energy,

$$\hat{i}^> > \frac{\hat{d}^> \hat{H}^>}{\hat{d}^> \hat{t}^>} = \hat{H} \times ({}^<\hat{T} - \hat{T}^>) \times \hat{H}. \quad (4.42).$$

Therefore, *the genotopic branch of hadronic mechanics is the only known formulation permitting nonconserved quantities to be Hermitean, thus being observability.* In fact, since they are Hamiltonian, other formulation attempt to represent nonconservation, e.g., by adding an "imaginary potential" to the Hamiltonian. In this case the Hamiltonian is non-Hermitean and, consequently, the nonconservation of the energy cannot be an observable. besides, said "nonconservative models" with non-Hermitean Hamiltonians are nonunitary and are formulated on conventional spaces over conventional fields, thus suffering all the catastrophic inconsistencies of Theorem 3.1. For additional aspects of genomechanics interested readers may consult Refs. [2,5].

4.10: Invariance of genotheories. Recall that a fundamental Axiomatic feature of quantum mechanics is the invariance under its time evolution of numerical predictsions and physical law, which invariance is due to the unitary structure of the theory. However, quantum mechanics is reversible and can only represent conventional conservation laws.

As indicated earlier, the representation of irreversibility and nonconservation require the use of nonunitary time evolutions which, however, are afflicted by the catastrophic inconsistencies of Theorem 3.1.

The only resolution of such a basic impasse known to this author has been the achievement of invariance under nonunitarity and irreversibility via the use of genomathematics, provided that such genomathematics is applied to the totality of the formalism to avoid evident inconsistencies caused by mixing different mathematics for the selected physical problem.

Such an invariance was first achieved by R. M. Santilli in Ref. (44) of 1997 can be illustrated by reformulating any given nonunitary transform in the *genounitary form*

$$U = \hat{U} \times \hat{T}^{>1/2}, W = \hat{W} \times \hat{T}^{>1/2}, \tag{4.43a}$$

$$U \times W^{\dagger} = \hat{U} > \hat{W}^{\dagger} = \hat{W}^{\dagger} > \hat{U} = \hat{I}^{>} = 1/\hat{T}^{>}, \tag{4.43b}$$

and then showing that genounits, genoproducts, genoexponentiations, etc., are indeed invariant under the above genounitary transform in exactly the same way as conventional unit, products, exponentiations, etc. are invariant under unitary transforms,

$$\hat{I}^{>} \rightarrow \hat{I}^{>\prime} = \hat{U} > \hat{I}^{>} > \hat{W}^{\dagger} = \hat{I}^{>}, \tag{4.44a}$$

$$\begin{gathered} \hat{A} > \hat{B} \rightarrow \hat{U} > (A > B) > \hat{W}^{\dagger} = \\ = (\hat{U} \times \hat{T}^{>} \times A \times T^{>} \times \hat{W}^{\dagger}) \times (\hat{T}^{>} \times W^{\dagger})^{-1} \times \hat{T}^{>} \times \\ \times (\hat{U} \times \hat{T}^{>})^{-1} \times (\hat{U} \times T^{>} \times \hat{A} \times T^{>} \times \hat{W}^{>}) = \\ = \hat{A}' \times (\hat{U} \times \hat{T}^{>} \times \hat{W}^{\dagger})^{-1} \times \hat{B} = \hat{A}' \times \hat{T}^{>} \times B' = \hat{A}' > \hat{B}', etc. \end{gathered} \tag{4.44b}$$

from which all remaining invariances follow, thus resolving the catastrophic inconsistencies of Theorem 3.1.

Note the *numerical invariance of the genounit* $\hat{I}^{>} \rightarrow \hat{I^{>}}' = \hat{I}^{>}$, *of the genotopic element* $\hat{T}^{>} \rightarrow \hat{T^{>}}' = \hat{T}^{>}$, *and of the genoproduct* $> \rightarrow >' => $ which are necessary to have invariant numerical predictions.

4.11: Simple construction of genotheories. As it was the case for the isotopies, a simple method has been identified in Ref. (44) for the construction of genotheories from any given conventional, classical or quantum formulation. It consists in identifying such genounits as the product of *two* different nonunitary transforms

$$U \times U^\dagger \neq 1, W \times W^\dagger \neq 1, U \times W^\dagger = \hat{I}^>, W \times U^\dagger = ^< \hat{I} = (\hat{I}^>)^\dagger, \tag{4.45}$$

and subjecting the totality of quantities and their operations of conventional models to said dual transforms,

$$I \rightarrow \hat{I}^> = U \times I \times W^\dagger, I \rightarrow ^< \hat{I} = W \times I \times U^\dagger, \tag{4.46a}$$

$$a \rightarrow \hat{a}^> = U \times a \times W^\dagger = a \times \hat{I}^>, a \rightarrow ^< \hat{a} = W \times a \times U^\dagger = ^< \hat{I} \times a, \tag{4.46b}$$

$$a \times b \rightarrow \hat{a}^> > \hat{b}^> = U \times (a \times b) \times W^> = (U \times a \times W^\dagger) \times (U \times W^\dagger)^{-1} \times (U \times b \times W^\dagger), \tag{4.46c}$$

$$\partial/\partial x \rightarrow \hat{\partial}^> / \hat{\partial}^> \hat{x}^> = U \times (\partial/\partial x) \times W^\dagger = \hat{I}^> \times (\partial/\partial x), \tag{4.46d}$$

$$< \psi| \times |\psi > \rightarrow <^< \psi| > |\psi^> >= U \times (< \psi| \times |\psi >) \times W^\dagger, \tag{4.46e}$$

$$H \times |\psi > \rightarrow \hat{H}^> > |\psi^> >= (U \times H \times W^\dagger) \times (U \times W^\dagger)^{-1} \times (U \times \psi >), etc. \tag{4.46f}$$

As a result, any given conventional, classical or quantum model can be easily lifted into the above genotopic form.

Note that the above construction implies that *all conventional physical quantities acquire a well defined direction of time.* For instance, the correct genotopic formulation of energy, linear momentum, etc., is given by

$$\hat{H}^> = U \times H \times W^\dagger, \hat{p}^> = U \times p \times W^>, etc., \tag{4.47}$$

and this explains the reason for having represented in this section energy, momentum and other quantities with their arrow of time $>$. Such an arrow can indeed be omitted for notational simplicity, but only after the understanding of its existence.

4.12: Genorelativity. Another important implication of genomathematics is the construction of yet another lifting of special relativity, this time intended for the invariant characterization of irreversible classical and quantum processes, today known as *genorelativity.* This new relativity is characterized by the genotopies of: the background Euclidean topology (14); the Minkowski space (15); the Poincaré symmetry (29); the physical laws; etc. The geno-Galilean case is treated in monographs (52,53) which appeared prior to the advent of the genodifferential calculus(14). The relativistic case is oulined in Rdf. (29).

Regrettably, we cannot review genorelativity in details to avoid a prohibitive length. For the limited scope of this presentation it is sufficient to indicate that genospecial relativity can be constructed either from the isospecial relativity indicated in the preceding section via the lifting of the isounits into nonsymmetric forms, or by subjecting conventional special relativity to the dual noncanonical or nonunitary transform,ms as in Eqs. (4.45) and (4.46).

4.13: Experimental verifications and applications. The experimental validity of genotheories in classical mechanics is established by the direct representation of all non-conservative and irreversible Newtonian systems by Geno-Hamilton equations via a simple algebraic calculation of the term K in Eqs. (4.34c).

The experimental validity of genotheories in particle physics is established by the fact that all dissipative nuclear models represented via imaginary potentials in the Hamiltonian and other nonunitary theories can be *identically* reformulated in terms of the genotopic branch of hadronic mechanics, while preserving the representation of experimental data identically.

Above all, genomathematics and its related formulations have indeed achieve the objective for which they were built, namely, *an invariant representation of irreversibility at all levels, from Newton to quantum mechanics.* Such an objective can be achieved via the following main rules:

(i) Identify the classical origin of irreversibility in the contact nonpotential forces among extended particles, much along the historical teaching of Newton (1), Lagrange (2) and Hamilton (3);

(ii) Represent said nonpotential forces via real-valued, nowhere singular, non-symmetric genounits and construct a mathematics which is structurally irreversible for all reversible Hamiltonians in the sense indicated earlier;

(iii) Achieve identical reformulation (4,34) of Hamilton's equations with external terms with a consistent algebra in the brackets of the time evolution of Lie-admissible type according to Albert (7);

(iv) Complement the latter mechanics with the underlying genosymplectic geometry, permitting the mapping of the classical formulations into operator formulations preserving said Lie-admissible character; and

(v) Identify the origin of irreversibility in the most elementary layers of nature, such as elementary particles in their irreversible motion in the interior of stars.

Note that a requirement for the above rules is the *nonconservation* of the energy and other physical quantities, which is readily verified by the geno-Hamilton equations (4.34) due to the lack of anti-symmetric character of brackets (A, B) of the time evolution.

An important application of genomechanics has been done by J. Ellis *et al.* (122) who have shown that its Lie-admissibility provides an axiomatically consistent, direct representation of irrebersibility in interior quasares structures.

In closing, it is hoped that Lagrange's and Hamilton's legacy of representing irreversibility with the external terms in their analytic equations is seriously considered because it implies covering theories with momentous advances in mathematics and all quantitative sciences.

5. CONSTRUCTION OF HYPERMECHANICS FROM BIOLOGICAL SYSTEMS

5.1: The scientific unbalance caused by biological systems. In this author's opinion, by far the biggest scientific unbalance of the 20-th century has been the treatment of biological systems (herein denoting DNA, cells, organisms, etc.) via the mathematics developed for inanimate matter.

The unbalance is due to the fact that conventional mathematics and related formulations, such as quantum mechanics, are inapplicable for the treatment of biological systems for various reasons. To begin, biological events, such as the growth of an organism, are irreversible. Therefore, any treatment of biological systems via conventional reversible mathematics and related physical formulations cannot pass the test of time. Quantum mechanics is ideally suited for the treatment of the structure of the Hydrogen atom or of crystals. These systems are represented by quantum mechanics as being ageless. Recall also that quantum mechanics is unable to treat deformations because of incompatibilities with basic formulations, such as the group of rotations. Therefore, the *rigorous application of quantum mechanics to biological structures implies that all organisms from cells to humans are perfectly rigid and eternal.*

5.2: The multivalued complexity of biological systems. Even after achieving the invariant formulation of irreversibility outlined in the preceding section, it was easy to see that the underlying genomathematics remains insufficient for in depth studies of biological systems. Recent studies conducted by C. R. Illert (56) have pointed out that the *shape* of sea shells can certainly be represented via conventional mathematics, such as the Euclidean geometry. However, the latter is inapplicable for a representation of the *growth in time* of sea shells. Computer simulations have shown that the imposition to sea shell growth of conventional geometric axioms (e.g., those of the Euclidean or Riemannian geometries) implies the lack of proper growth, as expected because said geometries are strictly reversible, while the growth of sea shells is strictly irreversible.

The same studies by C. R. Illert (*loc. cit.*) have indicated the need of a mathematics which is not only structurally irreversible, but also *multi-dimensional.* As an example, C. R. Illert achieved a satisfactory representation of sea shells via the *doubling of the Euclidean reference axes,* namely, a geometry which appears to be six-dimensional.

A basic problem in accepting such a view is the lack of compatibility with our sensory perception. In fact, when holding sea shells in our hands, we can fully perceive their shape as well as their growth with our three Eustachian tubes. In particular, our senses are sufficiently sensitive to perceive deviations from the Euclidean space, as well as for the possible presence of curvature.

These occurrences pose a rather challenging problem, the achievement of a representation of the complexity of biological systems via the *most general possible mathematics* which is: (1) is structurally irreversible (as in the preceding section); (2) admits the deformation theory; (3) is invariant under the time evolution; (4) is multi-dimensional; and, last but not least, (5) is compatible with our sensory perception.

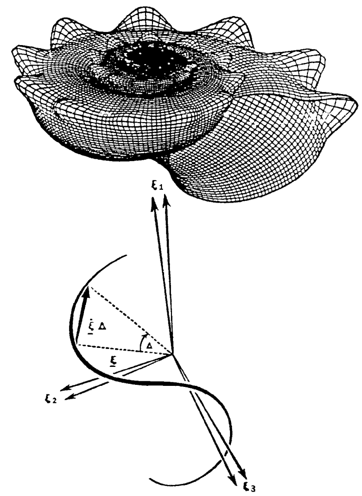

Figure 4: An illustration of the fourth scientific unbalance of the 20-th century, the representation of biological structures (including cells, organisms and their functions) via a mathematics which is not only Hamiltonian and reversible, but also single valued, while said biological systems require a mathematics which is not only nonhamiltonian and irreversible, but also multi-valued, as illustrated by the growth of a sea shell which, besides not being representable with a Hamiltonian and being irreversible, requires the doubling of all Euclidean reference axes.

A search in the mathematical literature revealed that a mathematics verifying all the above five requirements did not exist and had to be constructed from the main features of biological systems. As an example, *hyperstructures* in their current formulations (see Ref. (96) lack a well defined left and right generalized unit even under their *weak equalities*, they are not structurally irreversible, and lack invariance, thus not being suitable for applications in biology.

5.3: Elements of hypermathematics. After numerous trials and errors, a yet broader mathematics verifying the above five conditions was identified by R. M. Santilli in Ref. (14) (see also Refs. (13,47) monograph (57)); it is today known under the name of *hypermathematics;* and it is characterized by the following *hyperunits* here expressed for the lifting of the Euclidean unit

$$I = Diag.(1,1,1) \rightarrow \hat{I}^{>}(t,x,v,\psi,\partial_x\psi,...) = Diag.(\hat{I}_1^{>},\hat{I}_2^{>},\hat{I}_3^{>}) =$$
$$= Diag.[(\hat{I}_{11}^{>},\hat{I}_{12}^{>},...,\hat{I}_{1m}^{>}),(\hat{I}_{21}^{>},\hat{I}_{22}^{>},...,\hat{I}_{2m}^{>}),(\hat{I}_{31}^{>},\hat{I}_{32}^{>},...,\hat{I}_{3m}^{>})] = 1/\hat{T}^{>},$$
$$I = Diag.(1,1,...,1) \rightarrow^{<}\hat{I}(t,x,v,\psi,\partial_x\psi,...) = Diag.({}^{<}\hat{I}_1,{}^{<}\hat{I}_2,{}^{<}\hat{I}_3) =$$
$$= Diag.[({}^{<}\hat{I}_{11},{}^{<}\hat{I}_{12},...,{}^{<}\hat{I}_{1m}),({}^{<}\hat{I}_{21},{}^{<}\hat{I}_{22},...,{}^{<}\hat{I}_{2m}),{}^{<}(\hat{I}_{31},{}^{<}\hat{I}_{32},...,{}^{<}\hat{I}_{3m})] = 1/{}^{<}\hat{T},$$
$$\hat{I}^{>} = ({}^{<}\hat{I})^{\dagger}, \tag{5.1}$$

with corresponding *ordered hyperproducts to the right and to the left* as in Eqs. (4.3) and identifies as in Eqs. (4.4) expressed in terms of ordinary equalities and operations. A further broadening is then permitted by weak operations.

As one can see, the above mathematics *is not* $3m$*-dimensional, but rather it is 3-dimensional and m-multi-valued.* Such a feature permits the increase of the reference axes, e.g., for $m = 2$ we have the six axes used by C. R. Illert (*loc cit.*), while achieving compatibility with our sensory perception because at the abstract, realization-free level the hypermathematics characterized by hyperunit (5.1) is indeed 3-dimensional.

The various branches of hypermathematics (hypernumbers, hyperspaces, hyperalgebras, etc.) can be constructed via mere compatibility arguments with hyperunit (5.1) (see monograph (57) for brevity. The *isodual hypermathematics* can be constructed via the use of map (2.4).

A main difference of hypermathematics with the preceding formulations is that in the latter the product of two numbers is indeed generalized but single-valued, e.g., $2 > 3 = 346.765$. By comparison, in hypermathematics the product of two numbers yields, by conception, a *set of values,* e.g., $2 > 3 = (12.678, 341.329, 891.556, ...)$. Such a feature appears to be necessary for the representation of biological systems because the association of two atoms in a DNA (mathematically representable with the hypermultiplication) can yeld a large variety of different organs. This feature serves to indicate that the biological world has a complexity simply beyond our imagination at this time, and that the study of biological problems, such as understanding the DNA code via numbers dating back to biblical times, is manifestly insufficient.

An important application of hyperformulations has already been indicated in Section 3.14, the achievement of compatibility between structure models of hadrons with ordinayr massive particles as constituents in mutated conditions, and the $SU(3)$-color models of classification, which compatibility is permitted by the assumption that quarks are composite, thus admitting an hyperstructure with mutivalued hyperunits.

5.4: The complexity of hypertime and hyperrelativity for biological systems. The reader should be aware that the complexity of biological structures requires the use of hypermathematics as well as its isodual, e.g., for quantitative interpretations of bifurcations. In fact, a quantitative interpretation of bifurcations, e.g., in sea shells, requires *four different hypertimes and their isoduals,* as indicated below.

In turn, this is sufficient to illustrate the departure from conventional notions of a relativity suitable for quantitative studies on biological systems, known as *hyperrelativity and its isodual* (57). In fact, such new relativity requires the most general notion of numbers, those with a multi-valued hyperunits characterized by an ordered, yet unlimited number of non-Hermitean elements, with consequential most general possible geometries and mechanics, plus their isoduals. This results in an ordered, yet unlimited variety of spaces and their isoduals all coexisting in our three-dimensional Euclidean space, plus corresponding, equally co-existing varieties of time. There is little doubt that such features imply dramatic departures from the simplicity, thus insufficiency, of special relativity.

An illustration of the complexity of hyperformulations and corresponding hyperrelativity is given by the four different notions of hypertime which are needed for the description of complex biological processes, such as bifurcations in seashells, all in a coexisting form and each having a multi-valued character: motion forward in future time $\hat{t}^>$; motion backward in past time $^<\hat{t}$; motion forward from past time $^<\hat{t}^d$; and motion backward from future time $\hat{t}^{>d}$. The necessity of these four directions in time also illustrates the need of the isodual map.

5.5: Eric Trell's hyperbiological structures. A new conception of biological systems, which constitute a truly fundamental advance over rather simple prior conceptions, has been recently proposed by Erik Trell (see Ref. (164) and contributions quoted therein). It is based on representative blocks which appear in our space to be next to each other, thus forming a cell or an organism, while having in reality hypercorrelations, thus having the structure of hypernumbers, hypermathematics and hyperrelativity, with consequential descriptive capacities immensely beyond those of pre-existing, generally single-valued and reversible biological models. Regrettably, we cannot review Trell's new hyperbiological model to avoid an excessive length, and refer interested readers to the original literature (164).

SUMMARY

Mathematics	**Isodual mathematics**
Isomathematics	**Isodual isomathematics**
Genomathematics	**Isodual genomathematics**
Hypermathematics	**Isodual hypermathematics**
???	**Isodual ???**

Figure 5: A schematic view of the main advances outlined in this memoir, from which all applications uniquely follow, a progressive generalization of "mathematics" (here referred to the conventional formulation with left and right multiplicative unit defined over a field of characteristic zero) for the description of matter in conditions of increasing complexity and methodological needs, and their anti-isomorphic isoduals for corresponding progressive descriptions of antimatter. The last line illustrates the lack of final character of mathematics and, therefore, of scientific theories.

5.6: The lack of final character of all scientific theories. Main works of art, such as *Michelangelo's Pietá,* remain unsurpassed with the passing of time. By comparison, scientific theories, including mathematics, have a limited value in time, because, no matter how advanced and valid a given formulation may appear to be, the discovery of its structural generalizations is only a matter of time. Such a fate also holds for all generalized formulations outline in this memoir. In fact, the further broadening of hyperrelativity via the so-called *weak operations* is already within mathematical reach (13), the only missing elements being future discoveries sufficient to motivate their use.

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